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AN ELEMENTARY TREATISE

ON THE

DYNAMICS OF A SYSTEM OF RIGID BODIES.

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AN ELEMENTARY TREATISE

ON THE

DYNAMICS OF A SYSTEM OF RIGID BODIES.

With numerous Examples.

BY

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PREFACE.

In this Edition I have made several additions to each Chapter. I have tried, even at the risk of some little repetition, to make each Chapter, as far as possible, complete in itself, so that all that relates to any one part of the subject may be found in the same place. This arrangement is not only the most convenient for those already partly acquainted with the subject, but it will enable every student to select his own order in which to read the subject. The student who is just beginning Dynamics may not wish to be delayed by a long Chapter of preliminary analysis before he enters on the real subject of the book. He may therefore begin at D'Alembert's Principle, and only read those parts of Chapter I. to which reference is made. Some readers also may wish to pass on as soon as possible to the general principles of Angular Momentum and Vis Viva. Though a different order will be found advisable for different readers, I have ventured to indicate a course of reading for those who are just beginning the subject.

It will be observed that a Chapter has been devoted to the discussion of Motion in Two Dimensions. This course has been adopted because it seemed expedient to separate the difficulties of Dynamics from those of Solid Geometry.

The Examples which will be found at the end of each Chapter have been chiefly selected from the Examination Papers which have been set in the University and in the Colleges in the last few years.

I am desirous also of expressing how much I am indebted to Mr J. F. Moulton, B.A. Fellow of Christ's College, for the great assistance he has given me in correcting the proof sheets, and for the suggestions he has made to me. Many of the Examples also have been verified by him.

EDWARD J. ROUTH.

PETERHOUSE,
Sept. 10, 1868.

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The student to whom the subject is entirely new is advised to read *first* the following Articles: 1—17, 23, 36—89, 96—101, 127—135, 150—162, 183—196.

ERRATA.

Page	Line	
13	6	*For* YZ, ZX, XY *read* $-2YZ, -2ZX, -2XY$
28	19	*For* rectangular *read* similar
34	19	*For* any other point O *read* any other point P
71	13	*For* $1+at$ *read* $1+\alpha t$
95	19	*For* $\frac{dy}{dt}$ *read* $\frac{dy'}{dt}$
118	18	*For* eel *read* reel
127	28	*For* $\frac{1}{2}(xx''+x'^2)$ *read* $2(xx''+x'^2)$
170	26	*For* z *read* Z
171	3	*For* $\frac{dw}{dt}$ *read* $\frac{du}{dt}$
,,	14	*For* coordinates of L'' *read* coordinates of P''
,,	15	*For* the straight line OL'' *read* the straight line OP''
,,	24	*For* the line OL'' *read* the line OP''
185	14	*For* $x, y, z, \frac{dx}{dt}$ *read* $x, y, \frac{dx}{dt}$
196	29	*For* meet in a point *read* meet in some point O
203	25	*For* $(\Sigma mz'x_,)$ *read* $(\Sigma mz'x')$.
217	30	*For* $=$ *read* $-$
235	15	*For* $\frac{d}{dt}\Sigma mv^2$ *read* $\frac{1}{2}\frac{d}{dt}\Sigma mv^2$
245	6	*For* $d\theta$ *read* $\delta\theta$
259	9	*For* $\frac{dA}{dq}$ *read* $\frac{dH}{dq}$
263	9	*For* $\frac{dU_1}{dq_1}$ *read* $\frac{dU_1}{dq}$
265	19	*For* $(v'_z - v_z)$ *read* $(v'_z - v_z)v'_z$
288	7	*For* $-\frac{p^2}{c^2}L$ *read* $-\frac{p^2}{c^2}L_k$
298	8	*For* F_2 *read* E_2
,,	9	*For* F_3 *read* F_2

DYNAMICS OF A SYSTEM OF RIGID BODIES.

CHAPTER I.

ON MOMENTS OF INERTIA.

SECT. I. *Elementary Properties.*

1. DEF. 1. IF the mass of every particle of a material system be multiplied by the square of its distance from a straight line, the sum of the products so formed is called the *moment of inertia* of the system about that line.

DEF. 2. If M be the mass of a system, and κ be such a quantity that $M\kappa^2$ is its moment of inertia about a given straight line, then κ is called the *radius of gyration* of the system about that line.

DEF. 3. If two straight lines Ox, Oy be taken as axes, and if the mass of every particle of the system be multiplied by its *two* co-ordinates x, y, the sum of the products is called the *product of inertia* of the system about those two axes.

Let a body be referred to any rectangular axes, Ox, Oy, Oz, meeting in a point O, and let x, y, z be the co-ordinates of any particle m, then, according to these definitions, the moments of inertia about the axes of x, y, z respectively will be

$$\left.\begin{aligned} A &= \Sigma m\,(y^2+z^2) \\ B &= \Sigma m\,(z^2+x^2) \\ C &= \Sigma m\,(x^2+y^2) \end{aligned}\right\},$$

and the products of inertia about the axes of yz, zx, xy,

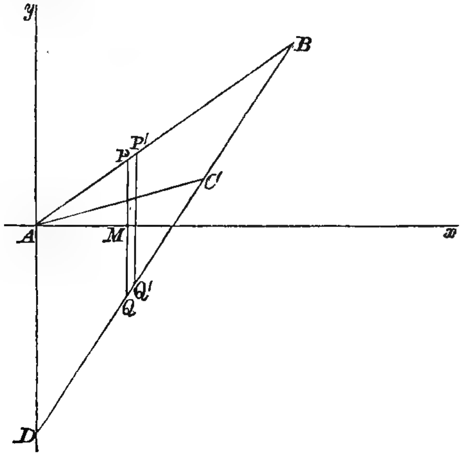
$$D = \Sigma m\,(yz),$$
$$E = \Sigma m\,(zx),$$
$$F = \Sigma m\,(xy).$$

From these definitions we see that the moments of inertia A, B, C are such that any two of them are together greater than the third.

2. It is clear that the process of finding moments and products of inertia is merely that of integration. We may illustrate this by the following example.

To find the moment of inertia of a uniform triangular plate about an axis in its plane passing through one angular point.

Let ABC be the triangle, Ay the axis about which the moment is required. Draw Ax perpendicular to Ay and produce BC to meet Ay in D. The given triangle ABC may be regarded as the difference of the triangles ABD, ACD. Let us then first find the moment of inertia of ABD. Let $PQP'Q'$ be an elementary area whose sides PQ, $P'Q'$ are parallel to the base AD and let PQ cut Ax in M. Let β be the distance of the angular point B from the axis Ay, $AM = x$ and $AD = l$.



Then the elementary area $PQP'Q'$ is clearly $l\frac{\beta - x}{\beta}dx$, and its moment of inertia about Ay is $l\frac{\beta - x}{\beta}dx\,.\,x^2$. Hence the moment of inertia of the triangle ABD

$$= \int_0^\beta l\left(1 - \frac{x}{\beta}\right)x^2dx = l\frac{\beta^3}{12}.$$

Similarly if γ be the distance of the angular point C from the axis Ay, the moment of inertia of the triangle ACD is $l\frac{\gamma^3}{12}$. Hence the moment of inertia of the given triangle ABC is $\frac{l}{12}(\beta^3 - \gamma^3)$.

Now $\frac{1}{2}l\beta$ and $\frac{1}{2}l\gamma$ are the areas of the triangles ABD, ACD. Hence if Δ be the area of the triangle ABC, the moment of inertia of the triangle about the axis Ay is

$$\frac{\Delta}{6}(\beta^2 + \beta\gamma + \gamma^2).$$

3. When the body is a lamina the moment of inertia about an axis perpendicular to its plane is equal to the sum of the moments of inertia about any two rectangular axes in its plane drawn from the point where the former axis meets the plane.

For let the axis of z be taken as the normal to the plane, then we have if A, B, C be the moments of inertia about the axes

$$A = \Sigma my^2, \quad B = \Sigma mx^2, \quad C = \Sigma m(x^2 + y^2),$$

and therefore $$C = A + B.$$

We may apply this theorem to the case of the triangle. Let β', γ' be the distances of the points B, C from the axis Ax. Then the moment of inertia of the triangle about a normal to the plane of the triangle through the point A is

$$= \frac{\Delta}{6}(\beta^2 + \beta\gamma + \gamma^2 + \beta'^2 + \beta'\gamma' + \gamma'^2).$$

This may be reduced to the more convenient form

$$=\frac{\Delta}{3}\left\{\left(\frac{AB}{2}\right)^2+\left(\frac{AC}{2}\right)^2+AE^2\right\},$$

where E is the middle point of BC.

4. In the following chapters of this treatise there are certain moments of inertia which occur so frequently that it will be found convenient to collect them together for reference. The reader is therefore advised to commit to memory the following table.

The moment of inertia of

(1) A rectangle whose sides are $2a$ and $2b$

about an axis through its centre in its plane perpendicular to the side $2a$ $\Big\} = \text{mass}\,\frac{a^2}{3}$,

about an axis through its centre perpendicular to its plane $\Big\} = \text{mass}\,\frac{a^2+b^2}{3}$.

(2) An ellipse semi-axes a and b

about the major axis $a = \text{mass}\,\frac{b^2}{4}$,

.... minor axis $b = \text{mass}\,\frac{a^2}{4}$,

about an axis perpendicular to its plane through the centre $\Big\} = \text{mass}\,\frac{a^2+b^2}{4}$.

In the particular case of a circle of radius a, the moment of inertia about a diameter is mass $\frac{a^2}{4}$, and about a perpendicular to its plane through the centre mass $\frac{a^2}{2}$.

(3) An ellipsoid semi-axes a, b, c

about the axis $a = \text{mass}\,\frac{b^2+c^2}{5}$.

In the particular case of a sphere of radius a the moment of inertia about a diameter $= \text{mass}\,\frac{2}{5}a^2$.

(4) A right solid whose sides are $2a$, $2b$, $2c$

$$\left.\begin{array}{l}\text{about an axis through its centre perpendicular}\\ \text{to the plane containing the sides } b \text{ and } c\end{array}\right\} = \text{mass}\,\frac{b^2+c^2}{3}.$$

5. As the process for determining these moments of inertia is very nearly the same for all these cases, it will be sufficient to consider only two instances.

To determine the moment of inertia of an ellipse about the minor axis.

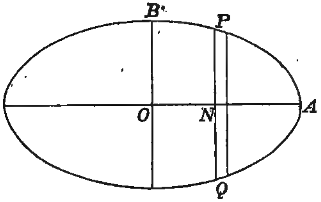
Let the equation to the ellipse be

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Take any elementary area PQ parallel to the axis of y, then clearly the moment of inertia is

$$4\mu\int_0^a x^2 y\,dx = 4\mu\,\frac{b}{a}\int_0^a x^2\sqrt{a^2 - x^2}\,dx,$$

where μ is the mass of a unit of area.



To integrate this, put $x = a\sin\phi$, then the integral becomes

$$a^4\int_0^{\frac{\pi}{2}} \cos^2\phi\sin^2\phi\,d\phi = a^4\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\phi}{8}\,d\phi = \frac{\pi a^4}{16};$$

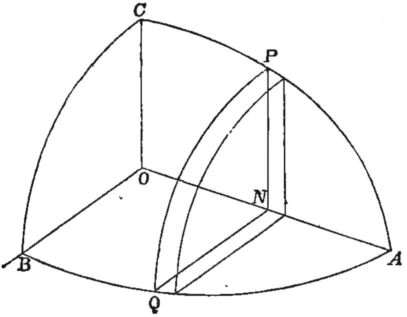
$$\therefore \text{ the moment of inertia} = \mu\pi ab\,\frac{a^2}{4} = \text{mass}\,\frac{a^2}{4}.$$

To determine the moment of inertia of an ellipsoid about a principal diameter.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Take any elementary area PNQ parallel to the plane of yz. Its area is evidently $\pi PN . QN$. Now PN is the value of z



when $y=0$, and QN the value of y when $z=0$, as obtained from the equation to the ellipsoid;

$$\therefore\ PN=\frac{c}{a}\sqrt{a^2-x^2},\quad QN=\frac{b}{a}\sqrt{a^2-x^2};$$

$$\therefore\ \text{the area of the element}=\frac{\pi bc}{a^2}(a^2-x^2).$$

Let μ be the mass of a unit of volume, then the whole moment of inertia

$$=\mu\int_{-a}^{a}\frac{\pi bc}{a^2}(a^2-x^2)\frac{PN^2+QN^2}{4}dx$$

$$=\mu\frac{\pi}{4}\frac{bc}{a^2}\int_{-a}^{a}(a^2-x^2)\frac{b^2+c^2}{a^2}(a^2-x^2)\,dx$$

$$=\mu\frac{4}{3}\pi abc\frac{b^2+c^2}{5}$$

$$=\text{mass}\,\frac{b^2+c^2}{5}.$$

6. If the mass of every particle of a material system be multiplied by the square of its distance from a given plane or from a given point, the sum of the products so formed is called the moment of inertia of the system with reference to that plane or point. If the given plane be taken as the plane of xy, then following the notation of Art. 1, the moment of inertia with respect to the plane is Σmz^2. If the given point be taken as the origin, the moment of inertia with respect to that point is Σmr^2.

If A, B, C be the moments of inertia about any three axes meeting at a point O, we have by Art. 1,

$$\begin{aligned} A+B+C &= 2\Sigma m\,(x^2+y^2+z^2) \\ &= 2\Sigma mr^2. \end{aligned}$$

Hence the sum of the moments of inertia about any three rectangular axes meeting at a given point is always the same and equal to twice the moment of inertia with respect to that point.

If J be the moment of inertia with respect to the plane xy, and I the moment of inertia about the axis of z, we have

$$I+J=\Sigma mr^2.$$

So that the sum of the moments of inertia of a system with reference to any plane through a given point and its normal at that point is constant and equal to the moment of inertia of the system with reference to that point.

7. It is possible to find by integration all moments and products of inertia, but the process may be simplified by the use of the two following propositions. The moments of inertia of a body about certain axes through its centre of gravity, which we may take as axes of reference, are given in the table of Art. 4. In order to find the moment of inertia about any other axis we shall investigate

(1) a method of comparing the required moment of inertia with that about a parallel axis through the centre of gravity;

(2) a method of determining the moment of inertia about this parallel axis in terms of the given moments of inertia about the axes of reference.

8. Prop. *Given the moments and products of inertia about all axes through the centre of gravity of a body, to deduce the moments and products about all other parallel axes.*

The moment of inertia of a body or system of bodies about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity plus the moment of inertia of the whole mass collected at the centre of gravity about the original axis.

The product of inertia about any two axes is equal to the product of inertia about two parallel axes through the centre of gravity plus the product of inertia of the whole mass collected at the centre of gravity about the original axes.

First, take the axis about which the moment of inertia is required as the axis of z. Let m be the mass of any particle of the body, which generally will be any small element.

Let x, y, z be the co-ordinates of m,

$\bar{x}, \bar{y}, \bar{z}$ those of the centre of gravity G of the whole system of bodies,

x', y', z' those of m referred to a system of parallel axes through the centre of gravity.

Then since $\dfrac{\Sigma mx'}{\Sigma m}, \dfrac{\Sigma my'}{\Sigma m}, \dfrac{\Sigma mz'}{\Sigma m}$

are the co-ordinates of the centre of gravity of the system referred to the centre of gravity as the origin, it follows that

$$\Sigma mx' = 0, \quad \Sigma my' = 0, \quad \Sigma mz' = 0.$$

The moment of inertia of the system about the axis of z is

$$\begin{aligned} &= \Sigma m\,(x^2 + y^2), \\ &= \Sigma m\,\{(\bar{x} + x')^2 + (\bar{y} + y')^2\}, \\ &= \Sigma m\,(\bar{x}^2 + \bar{y}^2) + \Sigma m\,(x'^2 + y'^2), \\ &\qquad + 2\bar{x}\,.\,\Sigma mx' + 2\bar{y}\,.\,\Sigma my'. \end{aligned}$$

Now $\Sigma m\,(\bar{x}^2 + \bar{y}^2)$ is the moment of inertia of a mass Σm collected at the centre of gravity, and $\Sigma m\,(x'^2 + y'^2)$ is the moment of inertia of the system about an axis through G, also $\Sigma mx' = 0$, $\Sigma my' = 0$; whence the proposition is proved.

Secondly, take the axes of x, y as the axes about which the product of inertia is required.

The product required is

$$\begin{aligned} &= \Sigma m\, xy, \\ &= \Sigma m\, (\bar{x} + x')\, (\bar{y} + y'), \\ &= \bar{x}\bar{y} \,.\, \Sigma m + \Sigma m\, (x'y') \\ &\qquad + \bar{x}\Sigma my' + \bar{y}\Sigma mx' \\ &= \bar{x}\bar{y}\Sigma m + \Sigma mx'y'. \end{aligned}$$

Now $\bar{x}\bar{y} \,.\, \Sigma m$ is the product of inertia of a mass Σm collected at G and $\Sigma mx'y'$ is the product of the whole system about axes through G; whence the proposition is proved.

9. Let there be two parallel axes A and B at distances a and b from the centre of gravity of the body. Then, if M be the mass of the material system,

$$\left.\begin{matrix}\text{moment of inertia}\\ \text{about } A\end{matrix}\right\} - Ma^2 = \left\{\begin{matrix}\text{moment of inertia}\\ \text{about } B\end{matrix}\right. - Mb^2.$$

Hence when the moment of inertia of a body about one axis is known, that about any other parallel axis may be found. It is obvious that a similar proposition holds with regard to the products of inertia.

10. The preceding proposition may be generalised as follows. Let any system be in motion, and let x, y, z be the coordinates at time t of any particle of mass m, then $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ are the velocities, and $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ the accelerations of the particle resolved parallel to the axes. Suppose

$$V = \Sigma m\phi\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \; y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \; z, \frac{dz}{dt}, \frac{d^2z}{dt^2}\right)$$

to be a given function depending on the structure and motion of the system, the summation extending throughout the system. Also let ϕ be an algebraic function of the first or second order. Thus ϕ may consist of such terms as

$$Ax^2 + Bx\frac{dy}{dt} + C\left(\frac{dz}{dt}\right)^2 + Eyz + Fx + \dots\dots\dots$$

where A, B, C, &c. are some constants. Then the following general principle will hold.

"The value of V for any system of co-ordinates is equal to the value of V obtained for a parallel system of co-ordinates with the centre of gravity for origin plus the value of V for the whole mass collected at the centre of gravity with reference to the first system of co-ordinates."

For let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity, and let $x = \bar{x} + x'$, &c. $\therefore \frac{dx}{dt} = \frac{d\bar{x}}{dt} + \frac{dx'}{dt}$, &c.

Now since ϕ is an algebraic function of the second order of x, $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$; y, &c. it is evident that on making the above substitution and expanding, the process of squaring &c. will lead to three sets of terms, those containing only $\bar{x}$, $\frac{d\bar{x}}{dt}$, $\frac{d^2\bar{x}}{dt^2}$, &c. those containing the products of $\bar{x}$, x', &c., and lastly those containing only x', $\frac{dx'}{dt}$, &c. The first of these will on the whole make up $\phi\left(\bar{x}, \frac{d\bar{x}}{dt}, \text{\&c.}\right)$, and the last $\phi\left(x', \frac{dx'}{dt}, \text{\&c.}\right)$.
Hence we have

$$V = \Sigma m\phi\left(\bar{x}, \frac{d\bar{x}}{dt} \dots\right) + \Sigma m\phi\left(x', \frac{dx'}{dt} + \dots\right)$$

$$+ \Sigma m\left(A\bar{x}\frac{dx'}{dt} + B\frac{d\bar{x}}{dt}x' + C\bar{x}\frac{dy'}{dt} + \dots\right),$$

where A, B, C, &c. are some constants.

Now the term $\Sigma m\left(\bar{x}\frac{dx'}{dt}\right)$ is the same as $\bar{x}\Sigma m\frac{dx'}{dt}$, and this vanishes. For since $\Sigma mx' = 0$, it follows that $\Sigma m\frac{dx'}{dt} = 0$. Similarly all the other terms in the second line vanish.

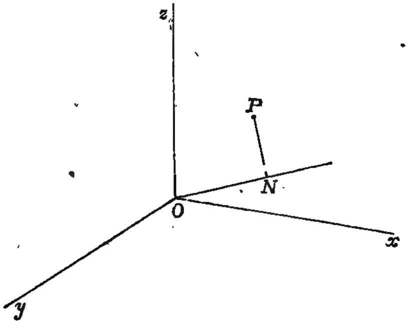
Hence the value of V is reduced to two terms. But the first of these is the value of V at the origin for the whole mass collected at the centre of gravity, and the second of these the value of V for the whole system referred to the centre of gravity as origin. Hence the proposition is proved.

The proposition would obviously be true if $\frac{d^3x}{dt^3}$, $\frac{d^3y}{dt^3}$, $\frac{d^3z}{dt^3}$, or any higher differential coefficients were also present in the function V.

11. Prop. *Given the moments and products of inertia about three straight lines at right angles meeting in a point, to deduce the moments and products of inertia about all other axes meeting in that point.*

Take these three straight lines as the axes of co-ordinates. Let A, B, C be the moments of inertia about the axes of x, y, z; D, E, F the products of inertia about the axes of yz, zx, xy. Let α, β, γ be the direction cosines of any straight line through the origin, then the moment of inertia I of the body about that line will be given by the equation

$$I = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta$$



Let P be any point of the body at which a mass m is situated, and let x, y, z be the co-ordinates of P. Let ON be the line whose direction cosines are α, β, γ, draw PN perpendicular to ON.

Since ON is the projection of OP, it is clearly

$$= x\alpha + y\beta + z\gamma,$$

also $$OP^2 = x^2 + y^2 + z^2,$$

and $$1 = \alpha^2 + \beta^2 + \gamma^2.$$

The moment of inertia I about ON

$$= \Sigma m\, PN^2$$

$$= \Sigma m\,\{x^2 + y^2 + z^2 - (\alpha x + \beta y + \gamma z)^2\}$$

$$= \Sigma m\,\{(x^2 + y^2 + z^2)(\alpha^2 + \beta^2 + \gamma^2) - (\alpha x + \beta y + \gamma z)^2\}$$

$$= \Sigma m\,(y^2 + z^2)\,\alpha^2 + \Sigma m\,(z^2 + x^2)\,\beta^2 + \Sigma m\,(x^2 + y^2)\,\gamma^2$$
$$- 2\Sigma myz\,.\,\beta\gamma - 2\Sigma mzx\,.\,\gamma\alpha - 2\Sigma mxy\,.\,\alpha\beta$$

$$= A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta.$$

12. This result may be exhibited geometrically. Let a radius vector OQ move in any manner about the given point O, and be of such length that the moment of inertia about OQ may be proportional to the inverse square of the length. Then if R represent the length of the radius vector whose direction-cosines are $(\alpha\beta\gamma)$, we have

$$I = \frac{\epsilon^4}{R^2},$$

where ϵ is some constant introduced to keep the dimensions correct, the mass being taken as unity. Hence the polar equation to the locus of Q is

$$\frac{\epsilon^4}{R^2} = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta.$$

Transforming to Cartesian co-ordinates, we have

$$\epsilon^4 = AX^2 + BY^2 + CZ^2 - 2DYZ - 2EZX - 2FXY,$$

which is the equation to a quadric. Thus to every point O of a material body there is a corresponding quadric which possesses the property that the moment of inertia about any radius vector is represented by the inverse square of that radius vector. The convenience of this construction is, that the relations which exist between the moments of inertia about straight lines meeting at any given point may be discovered by help of the known properties of a quadric. This quadric will be shown to be an ellipsoid, and is called the momental ellipsoid.

13. *If the momental ellipsoid be referred to any set of rectangular axes the coefficients of* X^2, Y^2, Z^2, YZ, ZX, XY *in its equation will be the moments and products of inertia about the axes.*

For let the equation to the surface be

$$P_1X^2 + P_2Y^2 + P_3Z^2 - 2Q_1YZ - 2Q_2ZX - 2Q_3XY = \epsilon^4.$$

Transforming to polar co-ordinates, we have

$$\frac{\epsilon^4}{R^2} = P_1\alpha^2 + P_2\beta^2 + P_3\gamma^2 - 2Q_1\beta\gamma - 2Q_2\gamma\alpha - 2Q_3\alpha\beta.$$

Let us compare this with the general expression for the moment of inertia about the line $(\alpha\beta\gamma)$ given in Art. 11. Then since the two results must be the same for all values of α, β, γ it is clear that we must have

$$P_1 = A, \quad P_2 = B, \quad P_3 = C,$$
$$Q_1 = D, \quad Q_2 = E, \quad Q_3 = F.$$

If the surface be referred to its principal diameters as axes of co-ordinates, its equation will be of the form

$$AX^2 + BY^2 + CZ^2 = \epsilon^4,$$

and A, B, C will be the moments of inertia about the new axes. Since A, B, C are essentially positive, it follows that the surface is an ellipsoid.

14. When three straight lines at right angles and meeting in a given point are such that if they be taken as axes of co-ordinates the products

$$\Sigma mxy, \quad \Sigma myz, \quad \Sigma mzx$$

all vanish, these are said to be Principal Axes at the given point.

The three planes through any two principal axes are called the Principal Planes at the given point.

The moments of inertia about the principal axes at any point are called the Principal Moments of Inertia at that point.

15. *At every point of a material system there are always three principal axes at right angles to each other.*

Construct the momental ellipsoid at the given point. Then it has been shown that the products of inertia about the axes are half the coefficients of XY, YZ, ZX in the equation to the momental ellipsoid referred to these straight lines as axes of co-ordinates. Now if an ellipsoid be referred to its principal diameters as axes, these coefficients vanish. Hence the principal diameters of the ellipsoid *are* the principal axes of the system. But every ellipsoid has at least three principal diameters, hence every material system has at least three principal axes.

If A, B, C be the three principal moments of inertia at any point, the expression given in Art. 11 for the moment of inertia about any other line becomes

$$I = A\alpha^2 + B\beta^2 + C\gamma^2.$$

16. The positions of the principal axes at many points in a body may also be found by inspection.

Thus the principal axes of an ellipsoid at the centre of gravity are its principal diameters. Let these be taken as axes, then the sum $\Sigma mxy = 0$. For if any element m be taken two of whose co-ordinates are x, y, another element m, of equal mass, can be found whose corresponding co-ordinates are $-x, y$. Hence the above sum for the whole body is zero. Similarly $\Sigma myz = 0$, $\Sigma mzx = 0$, and these, by Art. 14, are the conditions that the diameters are the principal axes at the centre.

So also the principal axes at the centre of an ellipse are the two principal diameters and a normal to the plane of the ellipse.

If the body be a lamina, one of the principal axes at any point O is a normal to the plane of the lamina. For take as axes of x, y any rectangular directions Ox, Oy in the plane of the lamina and let Oz be normal. Then since every element lies in the plane of xy we have $\Sigma mxz = 0$, $\Sigma myz = 0$, and hence the equation to the momental ellipsoid is

$$AX^2 + BY^2 + CZ^2 - 2DXY = \epsilon^4.$$

It is clear from the form of the equation that the axis of z is a principal diameter of the surface.

By a consideration of some simple properties of ellipsoids, the following propositions are evident:

I. Of the moments of inertia of a body about axes meeting at a given point, the moment of inertia about one of the principal axes is greatest and about another least.

For, in the momental ellipsoid, the moment of inertia about any radius vector from the centre is least when that radius vector is greatest and vice versa. And it is evident that the greatest and least radii vectores are two of the principal diameters.

II. If the three principal moments at any point O be equal to each other, the ellipsoid becomes a sphere. Every diameter is then a principal diameter, and the radii vectores are all equal. Hence every straight line through O is a principal axis at O, and the moments of inertia about them are all equal.

For example, the perpendiculars from the centre of gravity of a cube on the three faces are principal axes; for, the body being referred to them as axes, we clearly have $\Sigma mxy = 0$, $\Sigma myz = 0$, $\Sigma mzx = 0$. Also the three moments of inertia about them are by symmetry equal. Hence every axis through the centre of gravity of a cube is a principal axis, and the moments of inertia about them are all equal.

Next suppose the body to be a regular pyramid. Consider two planes drawn through the centre of gravity each parallel to a face of the pyramid. The relations of these two planes to the pyramid are in all respects the same. Hence also the momental ellipsoid at the centre of gravity must be similarly situated with regard to each of these planes, and the same is true for planes parallel to all the faces. Hence the ellipsoid must be a sphere and the moment of inertia will the same about every axis.

17. *To find the moments and products of inertia of a triangle about any axes whatever.*

If β and γ be the distances of the angular points B, C, of a triangle ABC from any straight line AX through the angle A, in the plane of the triangle, it is known that the moment of inertia of the triangle about AX is $\frac{\Delta}{6}(\beta^2 + \beta\gamma + \gamma^2)$, where Δ is the area of the triangle.

Let three equal particles, the mass of each being $\frac{\Delta}{3}$, be placed at the middle points of the three sides. Then it is easily seen, that the moment of inertia of the three particles about AX is the same as that of the triangle. The three particles treated as one system, and the triangle, have the same centre of gravity. Let this point be called O. Draw any straight line OX' through the common centre of gravity O parallel to AX, then it is evident that the moments of inertia of the two systems about OX' are also equal.

Since this equality exists for all straight lines through O in the plane of the triangle, it will be true for two straight lines OX', OY' at right angles, and therefore also for a straight line OZ' perpendicular to the plane of the triangle.

One of the principal axes at O of the triangle, and of the system of three particles is normal to the plane, and therefore the same for the two systems. The principal axes at O in the plane, are those two straight lines about which the moments of inertia are greatest and least, and therefore by what precedes these axes are the same for the two systems. If at any point two systems have the same principal axes and principal moments, they have also the same moments of inertia about all axes through that point, and the same products of inertia about any two straight lines meeting in that point. And if this point be the centre of gravity of both systems, the same thing will also be true for any other point.

If then a particle whose mass is one-third that of the triangle be placed at the middle point of each side, the moment of inertia of the triangle about any straight line, is the same as that of the system of particles, and the product of inertia about any two straight lines meeting one another, is the same as that of the system of particles about the same straight lines.

Three points D, E, F can always be found such that the products and moments of inertia of three equal particles placed at D, E, F, may be the same as the products and moments of inertia of any plane area. For let O be the centre of gravity of the area, Ox, Oy the principal axes at O in the plane of the area, and $M\alpha^2$ and $M\beta^2$ be the moments of inertia about these axes.

Let (xy), $(x'y')$, $(x''y'')$ be the co-ordinates of D, E, F, m the mass of a particle, so that $M = 3m$.

Then we must have

$$m\,(x^2 + x'^2 + x''^2) = M\beta^2,$$
$$m\,(y^2 + y'^2 + y''^2) = M\alpha^2,$$
$$xy + x'y' + x''y'' = 0.$$

Also, since the two systems must have the same centre of gravity

$$x + x' + x'' = 0,$$
$$y + y' + y'' = 0.$$

Eliminating $x'y'$, $x''y''$ from these equations, we get

$$\alpha^2x^2 + \beta^2y^2 = \tfrac{2}{3}\frac{M}{m}\,.\,\alpha^2\beta^2,$$
$$= 2\alpha^2\beta^2,$$

which is the equation to a momental ellipse. It easily follows, that D may be taken any where on this ellipse, and E and F are at the opposite extremities of that chord which is bisected in some point N by the produced radius DO, so that $ON = \frac{1}{2}\,OD$.

The momental ellipsoïd at the centre of gravity of any triangle may be found as follows.

Let an ellipse be inscribed in the triangle touching two of the sides AB, BC in their middle points F, D. Then, by Carnot's Theorem, it touches the third side CA in its middle point E. Since DF is parallel to CA the tangent at E, the straight line joining E to the middle point N of DF passes through the centre, and therefore the centre of the conic is the centre of gravity of the triangle.

This conic may be shown to be the momental ellipse of the triangle at O. To prove this, let us find the moment of inertia of the triangle about OE. Let $OE = r$, and let the semi-conjugate diameter be r', and ω the angle between r and r'. Now $ON = \frac{1}{2}r$, and hence from the equation to the ellipse

$$FN^2 = \tfrac{3}{4}r'^2,$$

$$\left.\begin{array}{r}\text{therefore moment of}\\ \text{Inertia about } OE\end{array}\right\} = \tfrac{2}{3}\Delta \cdot \tfrac{3}{4}r'^2 \sin^2\omega,$$

$$= \frac{\Delta}{2} \cdot \frac{\Delta'^2}{\pi^2 r^2};$$

where Δ' is the area of the ellipse, so that the moments of inertia of the system about OE, OF, OD are proportional inversely to OE^2, OF^2, OD^2. If we take a momental ellipse of the right dimensions, it will cut the inscribed conic in E, F, and D, and therefore also at the opposite ends of these diameters. But two conics cannot cut each other in six points unless they are identical. Hence this conic is the momental ellipse at O of the triangle.

A normal at O to the plane of the triangle is a principal axis of the triangle (Art. 16). Hence the momental ellipsoid of the triangle has the inscribed conic for one principal section. If a and b be the lengths of the axes of this conic, c that of the axis of the ellipsoid which is perpendicular to the plane of the lamina, we have by Arts. 3 and 12

$$\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

If the triangle be an equilateral triangle, the momental ellipsoid becomes a spheroid, and every axis through the centre of gravity in the plane of the triangle is a principal axis.

18. *To find the moments and products of inertia of a tetrahedron about any axes whatever.*

Let $ABCD$ be the tetrahedron. Through one angular point D draw any plane and let it be taken as the plane of

xy. Let D be the area of the base ABC; α, β, γ the distances of its angular points from the plane of xy, and p the length of the perpendicular from D on the base ABC.

Let PQR be any section parallel to the base ABC and of thickness du, where u is the perpendicular from D on PQR. The moment of inertia of the triangle PQR with respect to the *plane* of xy is the same as that of three equal particles, each one-third its mass, placed at the middle points of its sides. The volume of the element $PQR = \frac{u^2}{p^2} Ddu$. The ordinates of the middle points of the sides AB, BC, CA are respectively $\frac{\alpha+\beta}{2}$, $\frac{\beta+\gamma}{2}$, $\frac{\gamma+\alpha}{2}$. Hence, by similar triangles, the ordinates of the middle points of PQ, QR, RP are also $\frac{\beta+\gamma}{2}\frac{u}{p}$, $\frac{\gamma+\alpha}{2}\frac{u}{p}$, $\frac{\alpha+\beta}{2}\frac{u}{p}$.

The moment of inertia of the triangle PQR with regard to the plane xy is therefore

$$=\frac{1}{3}\frac{u^2}{p^2}Ddu\left\{\left(\frac{\beta+\gamma}{2}\frac{u}{p}\right)^2+\left(\frac{\gamma+\alpha}{2}\frac{u}{p}\right)^2+\left(\frac{\alpha+\beta}{2}\frac{u}{p}\right)^2\right\}.$$

Integrating from $u=0$ to $u=p$, we have the moment of inertia of the tetrahedron with regard to the plane xy

$$=\frac{V}{10}\{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta\},$$

where V is the volume.

If four-fifths of the mass of the tetrahedron were collected into one particle and placed at the centre of gravity of the tetrahedron, and one-twentieth at each of the angular points, the moment of inertia of these five particles with regard to the plane of xy would be

$$=V\frac{4}{5}\left(\frac{\alpha+\beta+\gamma}{4}\right)^2+\frac{V}{20}\alpha^2+\frac{V}{20}\beta^2+\frac{V}{20}\gamma^2,$$

which is the same as that of the tetrahedron.

The centre of gravity of these five particles is the centre of gravity of the tetrahedron, and they together make up the mass of the tetrahedron. Hence, by Art. 8, the moments of inertia of the two systems with regard to any plane through the centre of gravity are the same, and by the same article this equality will exist for all planes whatever. It follows by Art. 6, that the moments of inertia about any straight line are also equal. The two systems have therefore the same momental ellipsoids at the common centre of gravity. Hence also, by Art. 13, the products of inertia about any two straight lines are equal.

19. The reciprocal surface of the momental ellipsoid is another ellipsoid, which is called the Ellipsoid of Gyration. This second surface has also been employed to represent, geometrically, the position of the principal axes and the moment of inertia about any line.

To show that the reciprocal surface of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is the ellipsoid

$$a^2x^2+b^2y^2+c^2z^2=\epsilon^4.$$

Let ON be the perpendicular from the origin O on the tangent plane at any point P of the first ellipsoid, and let l, m, n be the direction-cosines of ON, then

$$ON^2=a^2l^2+b^2m^2+c^2n^2.$$

Produce ON to Q so that $OQ=\frac{\epsilon^2}{ON}$, then Q is a point on the reciprocal surface. Let $OQ=R$;

$$\therefore \frac{\epsilon^4}{R^2}=a^2l^2+b^2m^2+c^2n^2.$$

Changing this to rectangular co-ordinates, we get

$$\epsilon^4=a^2x^2+b^2y^2+c^2z^2.$$

It follows, therefore, that the principal diameters of the ellipsoid of gyration are the principal axes of the body, and the moment of inertia about the perpendicular on any tangent-plane is measured by the square of that perpendicular.

If ABC be the principal moments at the centre of gravity, the equation to the ellipsoid of gyration is

$$\frac{X^2}{A}+\frac{Y^2}{B}+\frac{Z^2}{C}=1.$$

It may be shown that the general equation to this ellipsoid referred to any set of rectangular axes is

$$\begin{vmatrix} A, & -F, & -E, & X \\ -F, & B, & -D, & Y \\ -E, & -D, & C, & Z \\ X, & Y, & Z & 1 \end{vmatrix}=0,$$

where A, B, C, D, E, F are the moments and products of inertia about the axes of co-ordinates.

20. There is a third ellipsoid which is sometimes used. By Arts. 6 and 11 the moment of inertia with reference to a plane whose direction-cosines are (α, β, γ) is

$$J=\frac{A+B+C}{2}-A\alpha^2-B\beta^2-C\gamma^2+2D\beta\gamma+2E\gamma\alpha+2F\alpha\beta$$

$$=\Sigma mx^2 . \alpha^2+\Sigma my^2 . \beta^2+\Sigma mz^2 . \gamma^2$$
$$+2\Sigma myz . \beta\gamma+2\Sigma mzx . \gamma\alpha+2\Sigma mxy . \alpha\beta,$$

if we follow the notation of Art. 1.

Hence, as in Art. 12, we may construct the ellipsoid

$$\left.\begin{array}{l}\Sigma mx^2 . X^2+\Sigma my^2 . Y^2+\Sigma mz^2 . Z^2 \\ +2\Sigma myz . YZ+2\Sigma mzx . ZX+2\Sigma mxy . XY\end{array}\right\}=\epsilon^4,$$

where the mass has been taken as unity. Then the moment of inertia with regard to any plane through the centre of the ellipsoid is represented by the inverse square of the radius vector perpendicular to that plane.

If we compare the equation of the momental ellipsoid with that of this third ellipsoid, we see that one may be obtained from the other by subtracting the same quantity from each of the coefficients of X^2, Y^2, Z^2. Hence the two ellipsoids have their circular sections coincident in direction.

This ellipsoid may also be used to find the moments of inertia about any straight line through the origin. For we may deduce from Art. 6 that the moment of inertia about any radius vector is represented by the difference between the inverse square of that radius vector and the sum of the inverse squares of the semi-axes.

The reciprocal of this ellipsoid has also been made use of. All these ellipsoids have their principal diameters coincident in direction, and any one of them may be used to determine the directions of the principal axes at any point.

21. *A straight line passes through a fixed point* O *and moves about it in such a manner that the moment of inertia about the line is always the same and equal to a given quantity* I. *To find the equation to the cone generated by the straight line.*

Let the principal axes at O be taken as the axes of co-ordinates, and let (α, β, γ) be the direction-cosines of the straight line in any position. Then by Art. 15 we have

$$A\alpha^2 + B\beta^2 + C\gamma^2 = I.$$

Hence the equation to the locus is

$$(A - I)\,\alpha^2 + (B - I)\,\beta^2 + (C - I)\,\gamma^2 = 0,$$

or, transforming to Cartesian co-ordinates,

$$(A - I)\,x^2 + (B - I)\,y^2 + (C - I)\,z^2 = 0.$$

It appears from this equation that the principal diameters of the cone are the principal axes of the body at the given point.

The given quantity I must be less than the greatest and greater than the least of the moments A, B, C. Let A, B, C be arranged in descending order of magnitude; then if I be less than B, the cone has its concavity turned towards the axis C, if I be greater than B the concavity is turned towards the axis A, if $I = B$ the cone becomes two planes which are coincident with the central circular sections of the momental ellipsoid at the point O.

The geometrical peculiarity of this cone is that its circular sections in all cases are coincident in direction with the circular sections of the momental ellipsoid whose centre is at the vertex.

This cone is called an equimomental cone at the point at which its vertex is situated.

22. The properties of products of inertia of a body about different sets of axes are not so useful as to require a complete discussion. The following theorems will serve as exercises.

(1) If any point O be given and any plane drawn through it, then two straight lines at right angles Ox, Oy can always be found such that the product of inertia about these lines is zero.

These are the axes of the section of the momental ellipsoid at the point O formed by the given plane.

(2) If two other straight lines at right angles Ox', Oy' be taken in the same plane making an angle θ measured in the positive direction with Ox, Oy respectively, then the product of inertia F' about Ox', Oy' is given by the equation

$$F' = \frac{1}{2} \sin 2\theta \, (A - B),$$

where A, B are the moments of inertia about Ox, Oy.

(3) If I be the moment of inertia about any line in this plane making an angle θ with Ox, then

$$I = A \cos^2 \theta + B \sin^2 \theta.$$

For the section of the momental ellipsoid by the plane is the ellipse whose equation is

$$Ax^2 + By^2 = \epsilon^4,$$

whence the property follows at once.

(4) Let $(\lambda\mu\nu)$ $(\lambda'\mu'\nu')$ be the direction-cosines of two straight lines Ox', Oy' at right angles passing through the origin O and referred to the principal axes at O as axes of co-ordinates. Then the product of inertia about these lines is

$$F' = \lambda\lambda' \Sigma mx^2 + \mu\mu' \Sigma my^2 + \nu\nu' \Sigma mz^2.$$

For let $(x'y'z')$ be the co-ordinates of any point (xyz) referred to Ox', Oy' and a third line Oz' as new axes of co-ordinates. Then

$$x' = \lambda x + \mu y + \nu z, \text{ and } y' = \lambda' x + \mu' y + \nu' z.$$

Hence, since $F' = \Sigma m x' y'$, the theorem follows by simple multiplication.

Since $\lambda\lambda' + \mu\mu' + \nu\nu' = 0$, we have

$$-F' = A\lambda\lambda' + B\mu\mu' + C\nu\nu'.$$

(5) If $(\lambda\mu\nu)$ be the direction-cosines of an axis Ox', then the direction-cosines $(\lambda'\mu'\nu')$ of another axis Oy' at right angles such that the product of inertia about Ox', Oy' is zero, are given by the equations

$$\frac{\lambda'}{(B-C)\mu\nu} = \frac{\mu'}{(C-A)\nu\lambda} = \frac{\nu'}{(A-B)\lambda\mu}.$$

For by (4) the equations to find $\lambda'\mu'\nu'$ are

$$\left.\begin{aligned} A\lambda\lambda' + B\mu\mu' + C\nu\nu' &= 0, \\ \lambda\lambda' + \mu\mu' + \nu\nu' &= 0, \end{aligned}\right\}$$

whence the theorem follows by cross multiplication.

By (1) this is equivalent to the geometrical theorem. Given a radius vector Ox' of an ellipsoid, find another radius vector Oy' such that Ox', Oy' are principal diameters of the section $x'Oy'$.

(6) Let (lmn) be the direction-cosines of any given straight line Oz', and let D', E' be the products of inertia about Oz', Oy'; Oz', Ox', where Ox', Oy' are any two straight lines at right angles. Then as Ox', Oy' turn round Oz', $D'^2 + E'^2$ is constant, and

$$D'^2 + E'^2 = (A-B)^2 (lm)^2 + (B-C)^2 (mn)^2 + (C-A)^2 (nl)^2.$$

For by (4),

$$-D' = Al\lambda + Bm\mu + Cn\nu,$$

$$-E' = Al\lambda' + Bm\mu' + Cn\nu';$$

$$\therefore D'^2 + E'^2 = A^2 l^2 (\lambda^2 + \lambda'^2) + 2ABlm (\lambda\mu + \lambda'\mu') + \&c.$$

But
$$\left.\begin{aligned}\lambda^2+\lambda'^2&=1-l^2=m^2+n^2,\\ \lambda\mu+\lambda'\mu'&=-lm,\end{aligned}\right\}$$
whence by substitution the theorem follows at once.

(7) If A', B' be the moments of inertia about Ox', Oy', then as Ox', Oy' turn round Oz', the value of $A'B'-F'^2$ is constant, and

$$A'B'-F'^2=BCl^2+CAm^2+ABn^2.$$

EXAMPLES.

1. The moment of inertia of an arc of a circle whose radius is a and which subtends an angle 2α at the centre

(*a*) about an axis through its centre perpendicular to its plane $=Ma^2$,

(*b*) about an axis through its middle point perpendicular to its plane $=2M\left(1-\dfrac{\sin\alpha}{\alpha}\right)a^2$,

(*c*) about the diameter which bisects the arc

$$=M\left(1-\frac{\sin 2\alpha}{2\alpha}\right)\frac{a^2}{2}.$$

2. The moment of inertia of the surface of a hemisphere of radius a and mass M about the diameter perpendicular to the base is $M\dfrac{2a^2}{3}$.

For, complete the sphere, then by Art. 6, the moment of inertia about any diameter is two-thirds of the moment of inertia with respect to the point.

3. If each element of the area of a triangle ABC be multiplied by the n^{th} power of its distance from a straight line through one angle A, then the sum of the products is

$$\frac{2\Delta}{(n+1)(n+2)}\,\frac{\beta^{n+1}-\gamma^{n+1}}{\beta-\gamma},$$

where β, γ are the distances of the angular points B, C from the straight line through the angular point A, and Δ is the area of the triangle.

4. The moment of inertia of a right cone of mass M about the axis is $M\frac{3}{10}b^2$. That about a straight line through the vertex perpendicular to the axis is $M\frac{3}{5}\left(a^2+\frac{b^2}{4}\right)$, and about a slant side $M\frac{3}{20}\frac{6a^2+b^2}{a^2+b^2}b^2$, where a is the altitude of the cone and b the radius of the base.

5. The moment of inertia of the part of the area of a parabola cut off by any ordinate at a distance x from the vertex is $M\frac{3}{7}x^2$ about the tangent at the vertex, and $M\frac{y^2}{5}$ about the principal diameter, where y is the ordinate corresponding to x.

6. The moment of inertia of the area of the lemniscate $r^2=a^2\cos 2\theta$ about a line through the origin in its plane and perpendicular to its axis is $M\frac{3\pi+8}{48}a^2$.

7. The moment of inertia of a triangle about a line perpendicular to its plane through its centre of gravity is $\frac{M}{36}(a^2+b^2+c^2)$, where a, b, c are the sides of the triangle.

8. The moment of inertia of an elliptic area about any diameter whose length is $2r$ is $\frac{a^2b^2}{4r^2}$. The momental ellipse at the centre of gravity of the area and lying in its plane is the ellipse itself.

9. If $k_1 k_2$ be the radii of gyration of an elliptic lamina about any two conjugate diameters

$$\frac{1}{k_1^{\,2}}+\frac{1}{k_2^{\,2}}=4\left(\frac{1}{a^2}+\frac{1}{b^2}\right).$$

10. Let three particles each one-sixth of the mass of an elliptic area be placed one at one extremity of the major axis and the other two at the extremities of the ordinate which bisects the semi-axis major, and let a fourth particle whose mass is one-half that of the area be placed at the centre of gravity. Then the moments and products of inertia of the system of four particles and of the elliptic area are the same for all axes whatever.

11. Find the moment of inertia of a heterogeneous ellipsoid about the major axis; the strata of equal density being similar concentric ellipsoids, and the density along the major axis varying as the distance from the centre. The result is $M\frac{2}{9}(b^2+c^2)$.

When the boundary of any heterogeneous body is a surface of equal density, proceed thus. First find the moment of inertia of a homogeneous body of density D bounded by any surface of equal density whose parameter is a. Let this moment be $D\phi(a)$. Then $D\phi'(a)da$ is the moment of inertia of a stratum of equal density. Replace D by the variable density ρ, then the moment of inertia required is $\int_0^a \rho\phi'(a)da$.

12. Find the moment of inertia of a heterogeneous ellipse about the minor axis, the strata of equal density being confocal ellipses, and the density along the minor axis varying as the distance from the centre. The result is

$$\frac{3M}{20}\,\frac{(a^5-c^5)-5a^3(a^2-c^2)}{(a^3-c^3)-3a(a^2-c^2)},$$

where a is the semi-major axis, and c half the distance between the foci.

13. Any product of inertia as D cannot numerically be so great as $\frac{A}{2}$.

14. If A, B, F be the moments and product of inertia of a lamina about any two rectangular axes in its plane, prove that AB is greater than F^2.

If t be any quantity, we have by Art. 1, $At^2+2Ft+B=\Sigma m(yt+x)^2=$ a positive quantity. Hence the roots of the quadratic $At^2+2Ft+B=0$ are imaginary.

15. If *ABC*, *DEF* be the moments and products of inertia about any rectangular axes meeting at a point *O*, prove that

$$(A+B-C)(B+C-A)>4E^2,$$

$$(A+B-C)(B+C-A)(C+A-B)>8DEF.$$

16. A lamina is bounded by four rectangular hyperbolas, two of them have the axes of co-ordinates for asymptotes, and the other two have the axes for principal diameters. Prove that the sum of the moments of inertia of the lamina about the co-ordinate axes is

$$\frac{1}{4}(\alpha^2-\alpha'^2)(\beta^2-\beta'^2);$$

where $\alpha\alpha'$, $\beta\beta'$ are the semi-major axes of the hyperbolas.

Take the equations $xy=u$, $x^2-y^2=v$, then the two moments of inertia are $A=\iint x^2J\,du\,dv$ and $B=\iint y^2J\,du\,dv$, where $\frac{1}{J}$ is the Jacobian of uv with regard to xy. This gives at once $A+B=\frac{1}{2}\iint du\,dv$, where the limits are clearly $u=\frac{\alpha^2}{2}$ to $\frac{\alpha'^2}{2}$, $v=\beta^2$ to $v=\beta'^2$.

17. A lamina is bounded on two sides by two similar ellipses, the ratio of the axes in each being m, and on the other two sides by two rectangular hyperbolas, the ratio of the axes in each being n. These four curves have their principal diameters along the co-ordinate axes. Prove that the product of inertia about the co-ordinate axes is $\dfrac{(\alpha^2-\alpha'^2)(\beta^2-\beta'^2)}{8(m^2+n^2)}$, where $\alpha\alpha'$, $\beta\beta'$ are the semi-major axes of the curves.

18. If two plane figures have the same momental ellipse at their centres of gravity, their projections will also.

Hence deduce that the momental ellipse of a triangle is the ellipse that touches the sides at their middle points.

SECT. II. *On the Positions of the Principal Axes of a System.*

23. PROP. *A straight line being given it is required to find at what point in its length it is a principal axis of the system, and if any such point exist to find the other two principal axes at that point.*

Take the straight line as axis of z, and any point O in it as origin. Let C be the point at which it is a principal axis, and let Cx', Cy' be the other two principal axes.

Let $CO = h$, $\theta =$ angle between Cx' and Ox. Then

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z - h \end{aligned}\right\}.$$

Hence

$$\left.\begin{aligned}\Sigma mx'z' = \cos\theta\,\Sigma mxz + \sin\theta\,\Sigma myz \\ - h\,(\cos\theta\,\Sigma mx + \sin\theta\,\Sigma my)\end{aligned}\right\} = 0 \ldots\ldots\ (1)$$

$$\left.\begin{aligned}\Sigma my'z' = -\sin\theta\,\Sigma mxz + \cos\theta\,\Sigma myz \\ - h\,(-\sin\theta\,\Sigma mx + \cos\theta\,\Sigma my)\end{aligned}\right\} = 0 \ldots\ldots\ (2)$$

$$\Sigma mx'y' = \Sigma m\,(y^2 - x^2)\,\frac{\sin 2\theta}{2} + \Sigma mxy\cos 2\theta = 0 \ldots\ldots\ldots\ (3)$$

The last equation shows that

$$\tan 2\theta = \frac{2\,\Sigma mxy}{\Sigma m\,(x^2 - y^2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$= \frac{2F}{B - A},$$

according to the previous notation.

The equations (1) and (2) must be satisfied by the same value of h. This gives as the condition that the axis of z is a principal axis at some point in its length,

$$\Sigma mxz\,\Sigma my = \Sigma myz\,\Sigma mx.$$

Substituting in (1) we get

$$h = \frac{\Sigma myz}{\Sigma my} = \frac{\Sigma mxz}{\Sigma mx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The equation (5) expresses the condition that the axis of z should be a principal axis at some point in its length; and the value of h gives the position of this point. The positions of the other two principal axes may then be found by equation (4).

To determine the geometrical meaning of this condition, take the plane of xz to pass through the centre of gravity of the body. Then we have $\Sigma my = 0$, and the equation becomes

$$\Sigma mx \, \Sigma myz = 0.$$

One of these factors must be zero. In order that h may be finite, we must have $\Sigma myz = 0$. Construct the momental ellipsoid at the centre of gravity. By Art. 12 its equation referred to axes of co-ordinates parallel to Cx, Cy, Cz, is

$$AX^2 + BY^2 + CZ^2 - 2EZX - 2FXY = \epsilon^4,$$

according to the previous notation. The coefficient D of YZ is zero, because by Art. 8

$$\Sigma myz = D + \Sigma m \, . \, \bar{y}\bar{z}.$$

The equation to a section parallel to the plane yz is

$$BY^2 + CZ^2 = \epsilon^4,$$

which is an ellipse referred to its principal diameters as axes. Hence, in order that a straight line may be a principal axis at some point not infinitely distant, it must be parallel to one of the principal diameters of the section of the momental ellipsoid at the centre of gravity, made by a plane perpendicular to the plane containing the axis and the centre of gravity.

If $\Sigma mxz = 0$ and $\Sigma myz = 0$, the equations (1) and (2) are both satisfied by $h = 0$. These are therefore the sufficient and necessary conditions that the axis of z should be a principal axis at the origin.

If the system be a plane lamina and the axis of z be a normal to the plane at any point, we have $z = 0$. Hence the conditions $\Sigma mxz = 0$ and $\Sigma myz = 0$ are satisfied. Therefore one of the principal axes at any point of a lamina is a normal to the plane at that point.

In the case of a surface of revolution bounded by planes perpendicular to the axis, the axis is a principal axis at any point of its length.

Again equation (4) enables us, when one principal axis is given, to find the other two. If $\theta = a$ be the first value of θ, all the others are included in $\theta = \alpha + n\frac{\pi}{2}$; hence all these values give only the same axes over again.

24. Since (4) does not contain h, it appears that if the axis of z be a principal axis at more than one point, the principal axes at those points are parallel. Again, in that case (5) must be satisfied by more than one value of h. But since h enters only in the first power, this cannot be unless

$$\Sigma mx = 0, \quad \Sigma my = 0,$$
$$\Sigma mxz = 0, \quad \Sigma myz = 0;$$

so that the axis must pass through the centre of gravity and be a principal axis at the origin, and therefore (since the origin is arbitrary) a principal axis at every point in its length.

If the principal axes at the centre of gravity be taken as the axes of x, y, z, (1) and (2) are satisfied for all values of h. Hence, if a straight line be a principal axis at the centre of gravity, it is a principal axis at every point in its length.

25. PROP. *Given the positions of the principal axes* Ox, Oy, Oz *at the centre of gravity* O, *and the moments of inertia about them, to find the positions of the principal axes at any point* P *in the plane of* xy, *and the moments of inertia about those axes.*

Let the mass of the body be taken as unity, and let A, B be the moments of inertia about the axes Ox, Oy, of which we shall suppose A the greater.

Take two points S and H in the axis of x on each side of the origin so that

$$OS = OH = \sqrt{A - B}.$$

Then because these points are in one of the principal axes at the centre of gravity, the principal axes at S and H are parallel to the axes of co-ordinates, and the moments of inertia about those in the plane of xy are respectively A and $B+\overline{OS}|^2=A$, and these being equal, any straight line through S or H in the plane of xy is a principal axis at that point, and the moment of inertia about it is equal to A.

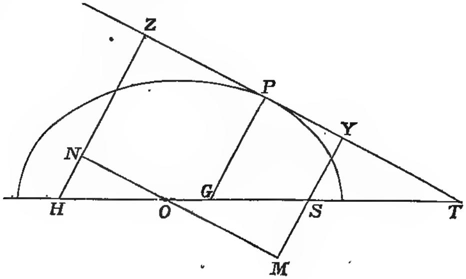
If P be any point in the plane of xy, then one of the principal axes at P will be perpendicular to the plane xy. For if p, q be the co-ordinates of P, the conditions that this line is a principal axis are

$$\left.\begin{array}{l}\Sigma m\,(x-p)\,z=0\\ \Sigma m\,(y-q)\,z=0\end{array}\right\},$$

which are obviously satisfied because the centre of gravity is the origin, and the principal axes the axes of co-ordinates.

The other two principal axes may be found thus. If two straight lines meeting at a point P be such that the moments of inertia about them are equal, then provided they are in a principal plane the principal axes at P bisect the angles between these two straight lines. For if with centre P we describe the momental ellipse, then the axes of this ellipse bisect the angles between any two equal radii vectores.

Join SP and HP; the moments of inertia about SP, HP are each equal to A. Hence, if PG and PT are the internal and external bisectors of the angle SPH; PG, PT are the principal axes at P. *If therefore with* S *and* H *as foci we describe any ellipse or hyperbola, the tangent and normal at any point are the principal axes at that point.*



26. Take any straight line MN through the origin, making an angle θ with the axis of x. Draw SM, HN perpendiculars on MN. The moment of inertia about it is

$$\begin{aligned} &= A\cos^2\theta + B\sin^2\theta \\ &= A - (A - B)\sin^2\theta \\ &= A - \overline{OS\sin\theta}|^2 \\ &= A - SM^2. \end{aligned}$$

Through P draw PT parallel to MN, and let SY and HZ be the perpendiculars from S and H on it. The moment of inertia about PT is then

$$\begin{aligned} &= \text{moment about } MN + MY^2 \\ &= A + (MY - SM)(MY + SM) \\ &= A + SY\,.\,HZ. \end{aligned}$$

In the same way it may be proved that the moment of inertia about a line PG passing between H and S is *less* than A by the product of the perpendiculars from S and H on PG.

If therefore with S *and* H *as foci we describe any ellipse or hyperbola, the moments of inertia about any tangent to either of these curves is constant.*

It follows from this that the moments of inertia about the principal axes at P are equal to $B + \left(\frac{SP \pm HP}{2}\right)^2$.

For if a and b be the axes of the ellipse we have

$$a^2 - b^2 = OS^2 = A - B,$$

and hence

$$A + SY\,.\,HZ = A + b^2 = B + a^2 = B + \left(\frac{SP + HP}{2}\right)^2,$$

and the hyperbola may be treated in a similar manner.

27. This reasoning may be extended to points lying in any given plane passing through the centre of gravity O of the body. Let Ox, Oy be the axes in the given plane such that the product of inertia about them is zero (Art. 22).

Construct the points S and H as before so that OS^2 and OH^2 are each equal to the difference of the moments of inertia about Ox and Oy. Draw Sy' a parallel through S to the axis of y, the product of inertia about Sx, Sy' is equal to that about Ox, Oy together with the product of inertia of the whole mass collected at O. Both these are zero, hence the momental ellipse at S is a circle, and the moment of inertia about every straight line through S in the plane xOy is the same and equal to that about Ox. We can then show that the moments of inertia about PH and PS are equal; so that PG, PT, the internal and external bisectors of the angle SPH are the principal diameters of the momental ellipse at P. And it also follows that the moments of inertia about the tangents to a conic whose foci are S and H are the same.

28. *Given the positions of the principal axes* OA, OB, OC *at the centre of gravity* O *and the moments of inertia* ABC *about them, to find the positions of the principal axes and the principal moments at any other point* O.*

Let a quadric be constructed confocal with the ellipsoid of gyration at O, and let the squares of its axes be $A+\lambda$, $B+\lambda$, $C+\lambda$, the mass of the body being taken as unity. Let OZ be a perpendicular from O on any plane drawn through P to touch this quadric, and let OZ cut in Y a parallel plane drawn to touch the ellipsoid of gyration. Let (lmn) be the direction cosines of the straight line OYZ, then

$$OZ^2 = (A+\lambda)\,l^2 + (B+\lambda)\,m^2 + (C+\lambda)\,n^2,$$

$$OY^2 = Al^2 + Bm^2 + Cn^2;$$

$$\therefore\ OZ^2 - OY^2 = \lambda.$$

$$\left.\begin{array}{r}\text{The moment of inertia about } OYZ\\ \text{by Art. 19}\end{array}\right\} = OY^2$$

$$= OZ^2 - \lambda;$$

$$\therefore \left.\begin{array}{r}\text{the moment of inertia about } PL\\ \text{a parallel to } OYZ \text{ through } P\end{array}\right\} = OZ^2 + PZ^2 - \lambda$$

$$= OP^2 - \lambda.$$

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* See two articles by Sir William Thomson and Mr Townsend, which appeared at the same time in the *Mathematical Journal*, 1846.

This is the same for all the tangent planes which can be drawn through P to touch any one confocal quadric. Hence the moment of inertia about any normal at P to the cone whose vertex is P and which envelopes any quadric confocal with the ellipsoid of gyration at O is constant and $= OP^2 - \lambda$.

These normals trace out the reciprocal cone of the enveloping cone, and it is proved in Solid Geometry that the principal diameters of a cone and its reciprocal cone are coincident in direction. Hence by Art. 21, the principal axes of the body at P are the principal diameters of the cone whose vertex is P and which envelopes any quadric confocal with the ellipsoid of gyration at the centre of gravity.

29. Let us consider the two extreme cases of this proposition.

First. Construct a quadric confocal with the ellipsoid of gyration and passing through P. The enveloping cone becomes a tangent plane, and one principal diameter is a normal to the quadric. Hence the three principal axes of the body at the point P are the three normals to the three quadrics which can be drawn through P confocal with the ellipsoid of gyration. Let λ_1, λ_2, λ_3 be the parameters of these three quadrics, then by what precedes, the principal moments are respectively $OP^2 - \lambda_1$, $OP^2 - \lambda_2$, $OP^2 - \lambda_3$.

If we wish to use only one quadric, we may consider the confocal ellipsoid through P. We know* that the normals

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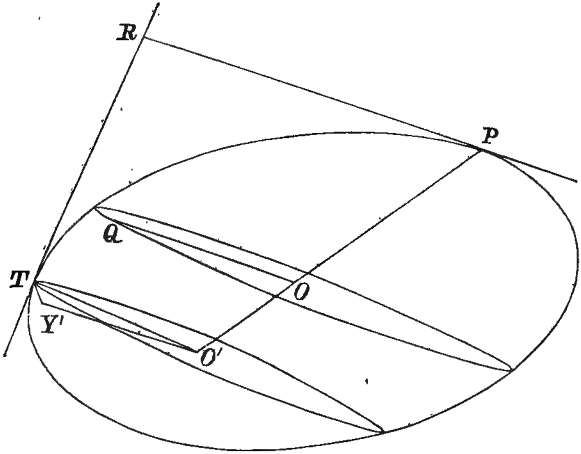
* These propositions are to be found in books on Solid Geometry, they may also be proved as follows.

Let the confocal ellipsoid pass near P and approach it indefinitely. The base of the enveloping cone is ultimately the Indicatrix; and as the cone becomes ultimately a tangent plane, one of its axes is ultimately a perpendicular to the plane of the Indicatrix. Now in any cone two of its axes are parallel to the principal diameters of any section perpendicular to the third axis. Hence the axes of the enveloping cone are the normal to the surface and parallels to the principal diameters of the Indicatrix. But all parallel sections of an ellipsoid are similar and similarly situated, hence the principal diameters of the Indicatrix are parallel to the principal diameters of the diametral section parallel to the tangent plane at P.

To find the principal moments, we may reason as follows. Let a tangent plane to the ellipsoid be drawn perpendicular to any radius vector OQ of the diametral section of OP, then the point of contact T, OQ and OP will lie in one plane when OQ is an axis of the section. For draw through T a section parallel to the other two confocals are tangents to the lines of curvature on the ellipsoid, and are also parallel to the principal diameters of the diametral section made by a plane parallel to the tangent plane at P. And if $D_1 D_2$ be these principal semi-diameters, we know that

$$\lambda_2 = \lambda_1 - D_1^2, \quad \lambda_3 = \lambda_1 - D_2^2.$$

to the diametral section, and let O' be its centre, and let $O'Y'$ be a perpendicular from O' on the tangent plane, which touches at T. Then OQ, $O'Y'$ and OP are in one plane. Now consider the section whose centre is O'; $O'Y'$ is the perpendicular on the tangent to an ellipse whose point of contact is T. Hence $O'Y'$, OT do not coincide unless $O'Y'$ be the direction of the axis of the ellipse. But this section is similar to the diametral section to which it was drawn parallel. Hence OQ is an axis of the diametral section.



Let PR be a straight line drawn through P parallel to OQ to meet in R the tangent plane which touches in T. Then RP, RT are two tangents at right angles to the ellipse PQT. Hence

$$\begin{aligned} OR^2 &= \text{sum of the squares of the semi-axes of the ellipse} \\ &= OP^2 + OQ^2 \end{aligned}$$

because OP, OQ are conjugate diameters.

The moment of inertia about PR, a perpendicular to a tangent plane, has been proved above to be $OR^2 - \lambda$, hence the moment of inertia about a parallel through P to the axis OQ is $OP^2 + OQ^2 - \lambda$.

Hence, if through any point P we describe the quadric

$$\frac{x^2}{A+\lambda}+\frac{y^2}{B+\lambda}+\frac{z^2}{C+\lambda}=1,$$

the axes of co-ordinates being the principal axes at the centre of gravity, then the principal axes at P are the normal to this quadric, and parallels to the axes of the diametral section made by a plane parallel to the tangent plane at P. And if these axes be $2D_1$ and $2D_2$, the principal moments at P are

$$OP^2-\lambda,\quad OP^2-\lambda+D_1^2,\quad OP^2-\lambda+D_2^2.$$

As we shall have to make frequent reference to the quadrics confocal with the ellipsoid of gyration at the centre of gravity, we shall call them the "subsidiary" quadrics.

30. *Secondly*. Let us consider the other extreme case of the proposition in Art. 28. Let the confocal quadric diminish until it become a focal conic, then the principal axes of a system at P are the principal diameters of a cone whose vertex is P and whose base is a focal conic of the ellipsoid of gyration at the centre of gravity.

The equi-momental cones at P are con-cyclic, hence their reciprocal cones have the same focal lines. These are the cones whose common vertex is P, and which envelope any subsidiary quadric. Thus the two cones whose vertex is P, and whose bases are the real focal conics are confocal and therefore also cut each other at right angles.

31. *The axes of co-ordinates being the principal axes at the centre of gravity it is required to express the condition that any given straight line may be a principal axis at some point in its length and to find that point.*

Let the equations to the given straight line be

$$\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}\;\dots\dots\dots\dots\dots\dots\;(1),$$

then it must be a normal to the quadric

$$\frac{x^2}{A+\lambda}+\frac{y^2}{B+\lambda}+\frac{z^2}{C+\lambda}=1\dots\dots\dots\dots\dots(2)$$

at the point at which the straight line is a principal axis.

Hence comparing the equation to the normal to (2) with (1), we have

$$\left.\begin{aligned}\frac{x}{A+\lambda}&=\mu l\\ \frac{y}{B+\lambda}&=\mu m\\ \frac{z}{C+\lambda}&=\mu n\end{aligned}\right\}\dots\dots\dots\dots\dots\dots\dots\dots(3),$$

these six equations must be satisfied by the same values of x, y, z, λ and μ.

Substituting for x, y, z from (3) in (1), we get

$$A\mu-\frac{f}{l}=B\mu-\frac{g}{m}=C\mu-\frac{h}{n},$$

eliminating μ from these last equations we have

$$\frac{\frac{f}{l}-\frac{g}{m}}{A-B}=\frac{\frac{g}{m}-\frac{h}{n}}{B-C}=\frac{\frac{h}{n}-\frac{f}{l}}{C-A}=\mu\dots\dots\dots\dots\dots(\text{E}).$$

This clearly amounts to only one equation, and is the required condition that the straight line should be a principal axis at some point in its length.

Substituting for x, y, z from (3) in (2), we have

$$\lambda\,(l^2+m^2+n^2)=\frac{1}{\mu^2}-(Al^2+Bm^2+Cn^2),$$

which gives one value only to λ. The values of λ and μ having been found, equations (3) will determine x, y, z, the co-ordinates of the point at which the straight line is a principal axis.

The geometrical meaning of this condition may be found by the following considerations, which were given by Mr Townsend in the *Mathematical Journal*. The normal and tangent plane at every point of a quadric will meet any principal plane in a point and a straight line, which are pole and polar with regard to the focal conic in that plane. Hence to find whether any assumed straight line is a principal axis or not, draw any plane perpendicular to the straight line and produce both the straight line and the plane to meet any principal plane at the centre of gravity. If the line of intersection of the plane be parallel to the polar line of the point of intersection of the straight line with respect to the focal conic, the axis will be a principal axis, if otherwise it will not be so. And the point at which the assumed straight line is a principal axis may be found by drawing a plane through the polar line perpendicular to the straight line. The point of intersection is the required point.

The analytical condition (E) exactly expresses the fact that the polar line is parallel to the intersection of the plane.

32. The following theorem is also due to the same author.

Let two points P, Q be so situated that a principal axis at P intersects a principal axis at Q. Then if two planes be drawn at P and Q perpendicular to these principal axes, their intersection will be a principal axis at the point where it is cut by the plane containing the principal axes at P and Q.

For let the principal axes at P, Q meet any principal plane at the centre of gravity in p, q, and let the perpendicular planes cut the same principal plane in LN, MN. Also let the perpendicular planes intersect each other in RN. Then RN is perpendicular to the plane containing the points P, Q, p, q. Also since the polars of p and q are LN, MN, it follows that pq is the polar of the point N. Hence the straight line RN satisfies the criterion of the last Article.

33. *To find the locus of the points at which two principal moments of inertia are equal to each other.*

The principal moments at any point P are

$$I_1 = OP^2 - \lambda, \quad I_2 = OP^2 - \lambda + D_1^2, \quad I_3 = OP^2 - \lambda + D_2^2.$$

If we equate I_1 and I_2 we have $D_1 = 0$, and the point P must lie on the elliptic focal conic of the ellipsoid of gyration.

If we equate I_2 and I_3 we have $D_1 = D_2$, so that P is an umbilicus of any ellipsoid confocal with the ellipsoid of gyration. The locus of these umbilici is the hyperbolic focal conic.

In the first of these cases we have $\lambda = -C$, and D_2 is the semi-diameter of the focal conic conjugate to OP.

Hence $D_2^2 + OP^2 =$ sum of squares of semi-axes

$$= A - C + B - C.$$

The three principal moments are therefore

$$\left.\begin{aligned} I_1 = I_2 &= OP^2 + C, \\ I_3 &= A + B - C, \end{aligned}\right\}$$

and the axis of unequal moment is a tangent to the focal conic.

The second case may be treated in the same way by using a confocal hyperboloid, we therefore have

$$\left.\begin{aligned} I_2 = I_3 &= OP^2 + B, \\ I_1 &= A + C - B, \end{aligned}\right\}$$

and the axis of unequal moment is a tangent to the focal conic.

34. *To find the curves on any subsidiary quadric at which the principal moments of inertia are equal to a given quantity* I.

First. All those points at which the principal moment about the normal is constant are found by making I_1 equal to the given quantity. This gives OP constant, and hence the required curve is the intersection of a sphere and a quadric. This curve is called a sphero-conic.

Secondly. All those points at which the principal moment about a tangent is constant are found by making I_2 or I_3 equal to the given quantity. Let r be the distance OP, a, b, c be the semi-axes of the quadric, and p the length of the perpendicular from the centre of gravity on the tangent plane at P. Then

$$\left.\begin{aligned} D_1 D_2 p &= abc, \\ r^2 + D_1^2 + D_2^2 &= a^2 + b^2 + c^2. \end{aligned}\right\}$$

Hence
$$I_2 = r^2 + D_1^2 - \lambda$$
$$= a^2 + b^2 + c^2 - \lambda - \frac{a^2b^2c^2}{p^2D_1^2}.$$

We know that along any line of curvature pD_1 is constant. *Hence the moments of inertia about the tangents to any line of curvature are equal to each other. And these tangents are principal axes at the points of contact.*

In the plane that contains I_2, I_3, and through their point of intersection draw any straight line making an angle ϕ with the axis I_2. This straight line is not in general a principal axis. The moment of inertia about it, is

$$I = I_1 \cos^2\phi + I_2 \sin^2\phi$$
$$= r^2 - \lambda + D_1^2 \cos^2\phi + D_2^2 \sin^2\phi$$
$$= r^2 + D_1^2 + D_2^2 - \lambda - D_1^2 D_2^2 \left(\frac{\cos^2\phi}{D_1^2} + \frac{\sin^2\phi}{D_2^2}\right)$$
$$= a^2 + b^2 + c^2 - \lambda - \frac{a^2b^2c^2}{p^2D^2},$$

where D is the radius vector parallel to the axis I of the section of the quadric made by a plane drawn through the centre parallel to the tangent plane at P. But we know that along any geodesic line pD is constant. *Hence the moments of inertia about the tangents to any geodesic line on the quadric are equal to each other.* These tangents are not in general principal axes at the points of contact.

35. The locus of all those points at which one of the principal moments of inertia of the body is constant is called an equi-momental surface.

To find the equation to such a surface we have only to put I_1 constant, this gives $\lambda = r^2 - I$. Substituting in the equation to the subsidiary quadric, the equation to the surface becomes

$$\frac{x^2}{x^2+y^2+z^2+A-I} + \frac{y^2}{x^2+y^2+z^2+B-I} + \frac{z^2}{x^2+y^2+z^2+C-I} = 1.$$

Through any point P on an equi-momental surface describe the subsidiary quadric such that the principal axis is a tangent to a line of curvature on the quadric. By Art. 34 one of the intersections of the equi-momental surface and this quadric is the line of curvature. Hence the principal axis at P about which the moment of inertia is I is a tangent to the equi-momental surface.

Again, construct the subsidiary quadric through P such that the principal axis is a normal at P, then one of the intersections of the momental surface and this quadric is the sphero-conic through P. The normal to the quadric, being the principal axis, has just been shown to be a tangent to the surface. Hence the tangent plane to the equi-momental surface, is the plane which contains the normal to the quadric and the tangent to the sphero-conic.

To draw a perpendicular from the centre O on this tangent plane, we may follow Euclid's rule. Take PP' a tangent to the sphero-conic, drop a perpendicular from O on PP', this is the radius vector OP, because PP' is a tangent to the sphere. At P in the tangent plane draw a perpendicular to PP', this is the normal PQ to the quadric. From O drop a perpendicular OQ on this normal, then OQ is a normal to the tangent plane. Hence this construction,

If P *be any point on an equi-momental surface whose parameter is* I *and* OQ *a perpendicular from the centre on the tangent plane, then* PQ *is the principal axis at* P *about which the moment of inertia is the constant quantity* I.

The equi-momental becomes Fresnel's wave surface when I is greater than the greatest principal moment of inertia at the centre of gravity. The general form of the surface is too well known to need a minute discussion here. It consists of two sheets, which become a concentric sphere and a spheroid when two of the principal moments at the centre of gravity are equal. When the principal moments are unequal, there are two singularities in the surface.

(1) The two sheets meet at a point P in the plane of the greatest and least moments. At P there is a tangent cone to the surface. Draw any tangent plane to this cone, and let OQ be a perpendicular from the centre of gravity O on this tangent plane. Then PQ is a principal axis at P. Thus

there are an infinite number of principal axes at P because an infinite number of tangent planes can be drawn to the cone. But at any given point, there cannot be more than three principal axes unless two of the principal axes be equal, and then the locus of the principal axes is a plane. Hence the point P is situated on a focal conic, and the locus of all the lines PQ is a normal plane to the conic. The point Q lies on a sphere whose diameter is OP, hence the locus of Q is a circle.

(2) The two sheets have a common tangent plane which touches the surface along a curve. This curve is a circle whose plane is perpendicular to the plane of greatest and least moments. Let OP' be a perpendicular from O on the plane of the circle, then P' is a point on the circle. If R be any other point on the circle the principal axis at R is RP'. Thus there is a circular ring of points at each of which the principal axis passes through the same point and the moments of inertia about these principal axes are all equal.

The equation to the equi-momental surface may also be used for the purpose of finding the three principal moments at any point whose co-ordinates (xyz) are given. If we clear the equation of fractions, we have a cubic to determine I whose roots are the three principal moments.

Thus let it be required to find the locus of all those points in a body at which any symmetrical function of the three principal moments is equal to a given quantity. We may express this symmetrical function in terms of the coefficients by the usual rules, and the equation to the locus is found.

EXAMPLES.

1. Determine the conditions that there may be a point in a body such that the moment of inertia about every axis through that point is the same.

Result. Two of the principal moments at the centre of gravity must be equal and each must be less than the third principal moment. There are then two points in the axis of unequal moment which satisfy the conditions.

2. The principal axes of a right-angled triangle at the right angle are, one perpendicular to the plane and two others inclined to its sides at the angles $\frac{1}{2}\tan^{-1}\frac{ab}{a^2-b^2}$, where a and b are the sides of the triangle adjacent to the right angle.

Take the formula $\tan 2\theta=\frac{2F}{B-A}$, Art. 23, then by Art. 17, $A=\Delta\frac{a^2}{6}$, $B=\Delta\frac{b^2}{6}$, $F=\Delta\frac{ab}{12}$.

3. The principal axes of a quadrant of an ellipse at the centre are, one perpendicular to the plane and two others inclined to the principal diameters at the angles $\frac{1}{2}\tan^{-1}\frac{4}{\pi}\frac{ab}{a^2-b^2}$, where a and b are the semi-axes of the ellipse.

4. The principal diameters of a cube at any point P are, the straight line joining P to O the centre of gravity of the cube, and any two straight lines at P perpendicular to PO.

5. Two particles each of mass m are placed at the extremities of the minor axis of an elliptic area of mass M. Prove that the principal axes at any point of the circumference of the ellipse will be the tangent and normal to the ellipse, provided $\frac{m}{M}=\frac{5}{8}\frac{e^2}{1-2e^2}$.

6. If P be any point in a principal plane at the centre of gravity, then every axis which passes through P, and is a principal axis at some point, lies in one of two perpendicular planes. One of these planes is the principal plane at the centre of gravity, and the other is a plane perpendicular to the polar line of P with regard to the focal conic.

7. If P be any point in a principal plane at the centre of gravity, then the locus of all the points Q at which QP is a principal axis is a circle passing through P and having its centre in the principal plane.

8. The edge of regression of the developable surface which is the envelope of the normal planes of any line of

curvature drawn on a subsidiary quadric is a curve such that all its tangents are principal axes at some point in each.

See Arts. 29 and 32.

9. The principal axes at the centre of gravity being the axes of reference, prove that the momental ellipsoid at the point (p, q, r) is

$$(A+q^2+r^2)X^2+(B+r^2+p^2)Y^2+(C+p^2+q^2)Z^2 \\ -2qrYZ-2rpZX-2pqXY=\epsilon^4,$$

when referred to its centre as origin.

10. Show that the cubic equation to find the three principal moments of inertia at any point (pqr) may be written in the form of a determinant

$$\begin{vmatrix} I-(A+q^2+r^2) & pq & rp \\ pq & I-(B+r^2+p^2) & qr \\ rp & qr & I-(C+p^2+q^2) \end{vmatrix}=0.$$

If (l, m, n) be proportional to the direction-cosines of the axis corresponding to any one of the values of I, their values may be found from the equations

$$\left.\begin{aligned} \{I-(A+q^2+r^2)\}l+pqm+rpn&=0, \\ pql+\{I-(B+r^2+p^2)\}m+qrn&=0, \\ rpl+qrm+\{I-(C+p^2+q^2)\}n&=0. \end{aligned}\right\}$$

11. If an equi-momental surface cut a quadric confocal with the ellipsoid of gyration at the centre of gravity, then the intersections are a sphero-conic and a line of curvature. But if the quadric be an ellipsoid, both these cannot be real.

For if the surface cut the ellipsoid in both, let P be a point on the line of curvature, and P' a point on the sphero-conic, then by Art. 29, $OP^2+D_1^2=OP'^2$, which is less than $A+\lambda$. But $OP^2+D_1^2+D_2^2=A+B+C+3\lambda$, therefore $D_2^2>B+C+2\lambda$, which is $>A+2\lambda$. Hence $D_2>$ than the greatest radius vector of the ellipsoid, which is impossible.

12. Find the locus of all those points in a body at which

(1) the sum of the principal moments is equal to a given quantity I.

(2) the sum of the products of the principal moments taken two and two together, is equal to I^2.

(3) the product of the principal moments is equal to I^3.

The results are

(1) a sphere whose radius is $\sqrt{\dfrac{I-(A+B+C)}{2M}}$, Art. 8.

(2) the surface

$$\left.\begin{array}{l}(x^2+y^2+z^2)^2+(A+B+C)(x^2+y^2+z^2)\\ +Ax^2+By^2+Cz^2+AB+BC+CA\end{array}\right\}=I^2, \text{ Art. 35.}$$

(3) the surface

$$A'B'C'-A'y^2z^2-B'z^2x^2-C'x^2y^2-2x^2y^2z^2=I^3,$$

where $A'=A+y^2+z^2$, with similar expressions for B', C'.

13. Prove that the locus of a point P at which one of the principal axes is parallel to a given straight line is a rectangular hyperbola in the plane of which the centre of gravity of the body lies, and one of the asymptotes is parallel to the given straight line. But if the given straight line be parallel to one of the principal axes at the centre of gravity, the locus of P is that principal axis or the perpendicular principal plane.

Take the origin at the centre of gravity, and the axis of x parallel to the given straight line.

CHAPTER II.

D'ALEMBERT'S PRINCIPLE, &c.

36. A RIGID body is a collection of material particles connected together by invariable geometrical relations. Our first attempt therefore to determine the motion of such a body would be to write down the equations of the several particles according to the principles laid down in treatises on Dynamics of a particle, and then to eliminate the unknown reactions between the particles, and thus obtain the equations of motion of a rigid body.

But if we attempt to do this, we are at once stopped by our ignorance of the nature of the actions of one particle on another. It would be necessary to make some assumptions in regard to these.

We might assume *first*, that the action between two particles is along the line which joins them; *secondly*, that the action and reaction between any two are equal and opposite.

The equations of motion of each separate particle on these assumptions may be easily written down. Suppose there are n particles, then there will be $3n$ equations, and as shown in any treatise on *Statics*, there must be at least $3n-6$, unknown reactions. It is therefore clear that after the elimination has been effected there cannot be *more than* 6 resulting equations free from the unknown reactions.

But if the equations are written down it will be seen that the reactions enter into the equation in such a manner that we can always eliminate them, however numerous the particles may be, and obtain six resulting equations.

But D'Alembert proposed a method by which these six resulting equations may be obtained without writing down the equations of motion of the several particles, and without making any assumption as to the nature of their mutual actions except the following:—

The internal actions and reactions of any system of rigid bodies in motion are in equilibrium amongst themselves.

37. *To explain D'Alembert's Principle.*[*]

Consider any particle of one of the rigid bodies of the system. Let m be its mass, (x, y, z) its co-ordinates at the time t referred to any fixed rectangular axes. Let F be the resultant of the impressed forces, R the resultant of the internal actions on the particle. The accelerations of the particle are $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$. Let f be the resultant of these accelerations.

Now if we supposed the particle separated from the rest of the body and acted on by the forces F and R, its acceleration would be f. But, as explained in Dynamics of a particle, the force which would produce this acceleration is measured by mf. Hence mf may be called the effective force on the particle m. And it follows that if we reverse this, the three F, R, mf are in equilibrium.

We may apply the same reasoning to every particle of the system, and we shall then have a group of forces similar to R, a group of forces similar to F, and a group similar to mf, acting on a group of particles connected by invariable relations, and these three groups will form a system of forces in equilibrium.

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[*] D'Alembert uses the following words:—"Soient A, B, C, &c. les corps qui composent le systême, et supposons qu'on leur ait imprimé les mouvemens a, b, c, &c. qu'ils soient forcés, à cause de leur action mutuelle, de changer dans les mouvemens a, b, c, &c. Il est clair qu'on peut regarder le mouvement a imprimé au corps A comme composé du mouvement a, qu'il a pris, et d'un autre mouvement α; qu'on peut de même regarder les mouvemens b, c, &c. comme composés des mouvemens b, β; c, γ; &c. d'où il s'ensuit que le mouvement des corps A, B, C, &c. entr'eux auroit été le même, si au lieu de leur donner les impulsions a, b, c, on leur eût donné à-la-fois les doubles impulsions a, α; b, β; &c. Or par la supposition les corps A, B, C, &c. ont pris d'eux-mêmes les mouvemens a, b, c, &c. donc les mouvemens α, β, γ, &c. doivent être tels qu'ils ne dérangent rien dans les mouvemens a, b, c, &c. c'est-à-dire que si les corps n'avoient reçu que les mouvemens α, β, γ, &c. ces mouvemens auroient dû se détruire mutuellement, et le systême demeurer en repos."

Now D'Alembert's principle assumes that the internal reactions between the particles of a system of rigid bodies are such that, taken all together, they form a system of forces in equilibrium. This is equivalent to asserting that the group R will itself form a system of forces in equilibrium. Whence it follows that the group F will be in equilibrium with the group mf.

Hence, *if forces equal to the effective forces but acting in exactly opposite directions were applied at each point of the system, these would be in equilibrium with the impressed forces.*

38. In D'Alembert's principle no assumption has been made as to the *nature* of the actions between the particles; hence the principle is true whether the particles be rigidly connected or not. We may, for example, apply the principle to the case of a fluid in motion or to any elastic or flexible body.

The principle is in reality an extension of the first law of motion. That law is equivalent to an assertion that the molecular actions of the particles which constitute a body do not affect the motion of translation of that body. D'Alembert's principle asserts further that they do not affect its motion when that motion consists of a combination of a motion of translation with one of rotation.

The truth of the principle cannot be established by abstract reasoning. It must be considered as resting on experimental evidence, or rather on that *inductive proof* which is derived from the accurate coincidence of the results of calculations founded on this principle with the observed motions of a rigid body.

39. *To apply D'Alembert's principle to obtain the equations of motion of a system of rigid bodies.*

Let (x, y, z) be the co-ordinates of the particle m at the time t referred to any set of rectangular axes fixed in space.

Then $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, and $\frac{d^2z}{dt^2}$, will be the accelerations of the particle. Let X, Y, Z be the impressed accelerating forces on the same particle resolved parallel to the axes. By D'Alembert's principle the forces

$$m\left(X-\frac{d^2x}{dt^2}\right),\quad m\left(Y-\frac{d^2y}{dt^2}\right),\quad m\left(Z-\frac{d^2z}{dt^2}\right),$$

together with similar forces on every particle will be in equilibrium. Hence by the principles of Statics we have the equation

$$\Sigma m\frac{d^2x}{dt^2}=\Sigma mX,$$

and two similar equations for y and z; these are obtained by resolving parallel to the axes. Also we have

$$\Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)=\Sigma m\,(yZ-zY),$$

and two similar equations for zx and xy; these are obtained by taking moments about the axes.

These equations may be written in the more convenient forms

$$\left.\begin{aligned}\frac{d}{dt}\Sigma m\frac{dx}{dt}&=\Sigma mX\\ \frac{d}{dt}\Sigma m\frac{dy}{dt}&=\Sigma mY\\ \frac{d}{dt}\Sigma m\frac{dz}{dt}&=\Sigma mZ\end{aligned}\right\}\text{.............. (A)},$$

$$\left.\begin{aligned}\frac{d}{dt}\Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)&=\Sigma m(yZ-zY)\\ \frac{d}{dt}\Sigma m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)&=\Sigma m(zX-xZ)\\ \frac{d}{dt}\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)&=\Sigma m(xY-yX)\end{aligned}\right\}\text{.......... (B)}.$$

In a precisely similar manner by taking the expressions for the accelerations in polar co-ordinates we should have obtained another but equivalent set of equations of motion.

40. Let us consider the meaning of these equations a little more minutely. If we multiply the mass m of any particle by its velocity v we have the momentum mv of that particle. Let us suppose the particle to be animated with this momentum acting in the direction of its motion. The sum of the resolved parts of the momenta of all the particles in any direction, and the sum of their moments about any straight line, will not be altered if we combine all these momenta together as if they were forces according to the rules of Statics. For the purpose therefore of finding these resolved parts and moments we may say that the whole momentum of a system is equivalent to a resultant linear momentum acting at any assumed point O, and a resultant couple. In the same way, if we multiply the mass of any particle by its acceleration f we have mf the effective force of the particle. These also may be compounded in the same way. So that the whole effective force of a body may be said to be equivalent to a resultant linear effective force acting at any assumed point O, and a resultant couple.

The momentum of any one particle m is equivalent to a single linear momentum acting at the origin O, and a couple. Let OM, ON be two straight lines drawn through the origin to represent in direction and magnitude at the time t the linear momentum and the axis of the couple. Let OM', ON' be the positions of these lines at the time $t+dt$. Then MM' and NN' represent the resultant linear momentum and the resultant couple added on in the time dt. Hence the *effective force* of any particle m is equivalent to a single linear effective force acting at O represented by $\frac{MM'}{dt}$, and a single effective couple whose axis is represented by $\frac{NN'}{dt}$.

Let OV, OH be two straight lines drawn through the origin O to represent in direction and magnitude the resultant linear momentum and resultant couple of the whole system at any time t. Let OV', OH' be the positions of these lines at the time $t+dt$. Then OV is the resultant of the group OM corresponding to all the particles of the system, and OV' the resultant of the group OM'. Hence $\frac{VV'}{dt}$ represents the whole linear *effective force* of the system at the time t. By similar reasoning $\frac{HH'}{dt}$ represents the resultant effective couple of the system. Thus it appears that the points V and H trace out two curves in space whose properties are analogous to those of the hodograph in Dynamics of a particle. From this reasoning it follows, that if V_x be the resolved part of the momentum of a system in the direction of any straight line Ox, and H_x the moment of the momentum about that straight line, then $\frac{dV_x}{dt}$ and $\frac{dH_x}{dt}$ are respectively the resolved part along, and the moment about that straight line, of the effective force of the whole system.

Let us now refer the whole system to Cartesian co-ordinates as in Art. 39. We see that $m\frac{dx}{dt}$, $m\frac{dy}{dt}$, $m\frac{dz}{dt}$ are the resolved parts of the momentum of the particle m. Hence OV is the resultant of $\Sigma m\frac{dx}{dt}$, $\Sigma m\frac{dy}{dt}$, and $\Sigma m\frac{dz}{dt}$. Also $m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$ is the moment of the momentum of the particle m about the axis of z. Hence OH is the resultant of

$$\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right),\quad \Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right),\quad \Sigma m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right).$$

Now D'Alembert's principle asserts that the whole effective forces of a system are together equivalent to the impressed forces. Hence whatever co-ordinates may be used, if X and L be the resolved parts and moment of the impressed moving forces respectively along and about any fixed straight line which we shall call the axis of x, the equations of motion are

$$\frac{dV_x}{dt}=X,\quad \frac{dH_x}{dt}=L.$$

The first of these corresponds to equations (A), the second to equations (B) of Art. 39.

We may notice the following cases.

(1) If no impressed forces act on the system, the two lines OV, OH are absolutely fixed in direction and magnitude throughout the motion.

(2) If all the impressed forces pass through a fixed point, let this point be chosen as the origin, then though OV may be variable, OH is fixed in position and magnitude.

(3) If all the impressed forces be equivalent to a system of couples, then though OH may be variable, OV is fixed in position and magnitude*.

41. The position of a body in space is given when we know the co-ordinates of some point in it and the angles which two straight lines fixed in the body make with the axes of co-ordinates. There are three geometrical relations existing between these six angles, so that the position of a body may be made to depend on *six* independent variables, viz. three co-ordinates and three angles. These might be called the *co-ordinates of the body*†. By the term "co-ordinates of a body" is meant any quantities which determine the position of the body in space.

It is evident that we may express the co-ordinates (x, y, z) of any particle m of a body in terms of the co-ordinates of that body and quantities which are known and remain constant during the motion. First, let us suppose the system to consist only of a single body, then if we substitute these expressions for x, y, z in the equations (A) and (B) of Art. 39, we shall have six equations to determine the six co-ordinates of the body in terms of the time. Thus the motion will be found. If the system consist of several bodies, we shall, by considering each separately, have six equations for each body. If there be any unknown reactions between the bodies, these will be included in X, Y, Z. For each reaction there will be a corresponding geometrical relation connecting the motion of those bodies. Thus on the whole we shall have sufficient equations to determine the motion of the system.

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* In a memoir on the differential coefficients and determinants of lines, Mr Cohen has discussed some of the properties of these resultant lines. *Phil. Trans.* 1862.

† Sir W. Hamilton uses the phrase "marks of position," but subsequent writers have adopted the term co-ordinates. See Cayley's *Report to the Brit. Assoc.* 1857.

In order that the equations (A) and (B) may not become complicated expressions after substitution for (x, y, z) in terms of the co-ordinates of the body, it is necessary to make a proper choice of what these co-ordinates shall be. In this we must be somewhat guided by our results.

42. *To find the resolved part of the momentum of a system in any direction.*

Let the given direction be taken as the axis of x. Let (x, y, z) be the co-ordinates of any particle whose mass is m. The resolved part of its momentum in the given direction is $m\frac{dx}{dt}$. Hence the resolved part of the momentum of the whole system is $\Sigma m\frac{dx}{dt}$. Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the centre of gravity of the system and M the whole mass. Then

$$M\bar{x} = \Sigma mx;$$

$$\therefore\ M\frac{d\bar{x}}{dt} = \Sigma m\frac{dx}{dt}.$$

Hence the resolved part of the momentum of a system in any direction is equal to the whole mass multiplied into the resolved part of the velocity of the centre of gravity.

That is, the linear momentum of a system is the same as if the whole mass were collected into its centre of gravity.

In the same way, the resolved part of the effective forces of a system in any direction is equal to the whole mass multiplied into the resolved part of the acceleration of the centre of gravity.

It appears from this proposition that it will be convenient to take the co-ordinates of the centre of gravity of each rigid body in the system as three of the co-ordinates of that body.

We can then express in a simple form the resolved part of the effective forces in any direction.

Having chosen the co-ordinates of the centre of gravity of a body as a means of expressing its linear momentum, it will generally be found convenient to refer its angular motion to a system of co-ordinate axes meeting at its centre of gravity. We shall therefore need the following proposition.

43. *The moment of the momentum of a system about any straight line is equal to the moment of the momentum of the whole mass supposed collected at its centre of gravity and moving with it, plus the moment of the momentum of the system relative to its centre of gravity about a straight line drawn parallel to the given straight line through the centre of gravity.*

Let the given straight line be taken as the axis of x, then following the same notation as before, the moment of the momentum about the axis of x is

$$\Sigma m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right).$$

Now this is an expression of the second degree. If then, we substitute $y = \bar{y} + y'$, $z = \bar{z} + z'$, we get by Art. 10

$$\Sigma m \left(y' \frac{dz'}{dt} - z' \frac{dy'}{dt} \right) + M \left(\bar{y} \frac{d\bar{z}}{dt} - \bar{z} \frac{d\bar{y}}{dt} \right),$$

where M is the mass of the system or body under consideration.

The second term of this expression is the moment about the axis of x of the momentum of a mass M moving with the centre of gravity.

The first term is the moment about a straight line parallel to the axis of x, not of the actual momenta of all the several particles but of their momenta *relatively* to that of the centre of gravity. In the case of any particular body it therefore depends only on the motion of the body relatively to its centre of gravity. In finding its value we shall suppose the centre of gravity reduced to rest by applying to every particle of the system a velocity equal and opposite to that of the centre of gravity.

In the same way, this proposition will be also true if for the "momentum" of the system we substitute "effective force."

44. We cannot at this stage find convenient expressions for the moment of the momentum about any straight line through the centre of gravity. Different expressions will be found advantageous under different circumstances. These will be discussed in the subsequent chapters of this treatise.

The quantity $\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$ expresses the moment of the momentum about the axis of z. It is then called the *angular momentum* of the system about the axis of z. There is another interpretation which can be given to it. If we transform to polar co-ordinates, we have

$$x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt}.$$

Now $\frac{1}{2}r^2d\theta$ is the elementary area described round the origin in the time dt by the projection of the particle on the plane of xy. If twice this polar area be multiplied by the mass of the particle, it is called the area conserved by the particle in the time dt round the axis of z. Hence

$$\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$$

is called the *area conserved* by the system in a unit of time, or more simply the area conserved.

45. We may now collect together for reference the results of the preceding articles.

Let u, v, w be the velocities of the centre of gravity of any rigid body of mass M resolved parallel to any three fixed rectangular axes, let h_1 h_2 h_3 be the three moments of the momentum relative to the centre of gravity about three rectangular axes fixed in direction and meeting at the centre of gravity. Then the effective forces of the body are equivalent to the three effective forces $M\frac{du}{dt}$, $M\frac{dv}{dt}$, $M\frac{dw}{dt}$

acting at the centre of gravity parallel to the directions into which the velocities have been resolved, and to the three effective couples $\frac{dh_1}{dt}$, $\frac{dh_2}{dt}$, $\frac{dh_3}{dt}$ about the axes meeting at the centre of gravity about which the moments were taken. The effective forces of all the other bodies of the system may be expressed in a similar manner.

Then all these effective forces and couples, being reversed, will be in equilibrium with the impressed forces. The equations of equilibrium may then be found by resolving in such directions and taking moments about such straight lines as may be most convenient. Instead of reversing the effective forces it is usually found more convenient to write the impressed and effective forces on opposite sides of the equations.

46. There is another mode of enunciating the results of arts 42 and 43 which is often useful.

(1) The motion of the centre of gravity of a system acted on by any forces is the same as if all the mass were collected at the centre of gravity and all the forces were applied at that point parallel to their former directions.

(2) The motion of a body acted on by any forces about its centre of gravity is the same as if the centre of gravity were fixed and the same forces acted on the body.

These are called the principles of the conservation of the motions of translation and rotation.

These two propositions follow at once from the summary of results given in Art. 45. It may however be useful to deduce them from first principles.

Taking any one of the equations (A) we have

$$\Sigma m \frac{d^2x}{dt^2} = \Sigma mX.$$

If $\bar{x}\ \bar{y}\ \bar{z}$ be the co-ordinates of the centre of gravity, then

$$\bar{x}\Sigma m = \Sigma mx$$

$$\therefore \frac{d^2\bar{x}}{dt} \Sigma m = \Sigma mX,$$

and the other equations may be treated in a similar manner.

But these are the equations which give the motion of a mass Σm acted on by forces ΣmX &c. Hence the first principle is proved.

Taking any one of equations (B) we have

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\Sigma m\,(xY-yX).$$

Let $x=\bar{x}+x'$, $y=\bar{y}+y'$, $z=\bar{z}+z'$, then by Art. 10 this equation becomes

$$\Sigma m\left(x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right)+\left(\bar{x}\frac{d^2\bar{y}}{dt^2}-\bar{y}\frac{d^2\bar{x}}{dt^2}\right)\Sigma m=\Sigma m\,(xY-yX).$$

Now the axes of co-ordinates are quite arbitrary, let them be so chosen that the centre of gravity is passing through the origin at the moment under consideration. Then $\bar{x}=0$, $\bar{y}=0$, but $\frac{d\bar{x}}{dt}$ $\frac{d\bar{y}}{dt}$ are not necessarily zero. The equation then becomes

$$\Sigma m\left(x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right)=\Sigma m\,(x'Y-y'X).$$

This equation does not contain the co-ordinates of the centre of gravity and holds at every separate instant of the motion and therefore is always true. But this and the two similar equations obtained from the other two equations of (B) are exactly the equations of moments we should have had if we had regarded the centre of gravity as a fixed point and taken it as the origin of moments.

Application of D'Alembert's Principle to impulsive forces.

47. If a force F act on a particle of mass m always in the same direction, the equation of motion is

$$m\frac{dv}{dt}=F,$$

where v is the velocity of the particle at the time t. Let T

be the interval during which the force acts, and let v, v' be the velocities at the beginning and end of that interval. Then

$$m(v' - v) = \int_0^T F dt.$$

Now suppose the force F to increase without limit, while the interval T decreases without limit. Then the integral may have a finite limit. Let this limit be P. Then the equation becomes

$$m(v' - v) = P.$$

The velocity in the interval T has increased or decreased from v to v'. Supposing the velocity to have remained finite, let V be its greatest value during this interval. Then the space described is less than VT. But in the limit this vanishes. Hence the particle has not moved during the action of the force F. It has not had time to move but its velocity is suddenly changed from v to v'.

We may consider that a proper measure has been found for a force when from that measure we can deduce all the effects of the force. In the case of finite forces we have to determine both the change of place and the change in the velocity of the particle. It is therefore necessary to divide the whole time of action into elementary times and determine the effect of the force during each of these. But in the case of infinite forces which act for an indefinitely short time, the change of place is zero, and the change of velocity is the only element to be determined. It is therefore more convenient to collect the whole force expended into one measure. Such a force is called an impulse. It may be defined as the limit of a force which is infinitely great, but acts only during an infinitely short time. There are of course no such forces in nature, but there are forces which are very great, and act only during a very short time. The blow of a hammer is a force of this kind. They may be treated as if they were impulses, and the results will be more or less correct according to the magnitude of the force and the shortness of the time of action. They may also be treated as if they were finite forces, and the displacement of the body during the time of action of the force may be found.

The quantity P may be taken as the measure of the force. An impulsive force is measured by the whole momentum generated by the impulse.

48. *In determining the effect of an impulse on a body, the effect of all finite forces which act on the body at the same time may be omitted.*

For let a finite force f act on a body at the same time as an impulsive force F. Then as before we have

$$v' - v = \frac{\int_0^T F dt}{m} + \frac{\int_0^T f dt}{m}$$

$$= \frac{P}{m} + \frac{fT}{m}.$$

But in the limit fT vanishes. Similarly the force f may be omitted in the equation of moments.

49. *To obtain the general equations of motion of a system acted on by any number of impulses at once.*

Let u, v, w, u', v', w' be the velocities of a particle of mass m parallel to the axes just before and just after the action of the impulses. Let X', Y', Z' be the resolved parts of the impulse on m parallel to the axes.

Taking the same notation as before, we have the equation

$$\Sigma m \frac{d^2x}{dt^2} = \Sigma m X,$$

or integrating

$$\Sigma m (u' - u) = \Sigma m \int_0^T X dt$$

$$= \Sigma X' \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad (1).$$

Similarly we have the equations

$$\Sigma m (v' - v) = \Sigma Y' \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad (2),$$

$$\Sigma m (w' - w) = \Sigma Z' \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad (3).$$

Again the equation

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\Sigma m\,(xY-yX)$$

becomes on integration

$$\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\Sigma m\left(x\int Ydt-y\int Xdt\right),$$

or taken between limits,

$$\Sigma m\,\{x\,(v'-v)-y\,(u'-u)\}=\Sigma\,(xY'-yX')\ldots\ldots\ldots(4),$$

and the other two equations become

$$\Sigma m\,\{y\,(w'-w)-z\,(v'-v)\}=\Sigma\,(yZ'-zY')\ldots\ldots\ldots(5),$$

$$\Sigma m\,\{z\,(u'-u)-x\,(w'-w)\}=\Sigma\,(zX'-xZ')\ldots\ldots\ldots(6).$$

In all the following investigations it will be found convenient to use accented letters to denote the states of motion after impact which correspond to those denoted by the same letters unaccented before the action of the impulse. Since the changes in direction and magnitude of the velocities of the several particles of the bodies are the only objects of investigation, it will be more convenient to express the equations of motion in terms of these velocities, and to avoid the introduction of such symbols as $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$.

50. In applying D'Alembert's Principle to impulsive forces the only change which must be made is in the mode of measuring the effective forces. If (u, v, w), (u', v', w') be the resolved parts of the velocity of any particle, just before and just after the impulse, and if m be its mass, the effective forces will be measured by $m\,(u'-u)$, $m\,(v'-v)$, and $m\,(w'-w)$. The quantity mf in Art. 37 is to be regarded as the measure of the impulsive force which, if the particle were separated from the rest of the body, would produce these changes of momentum.

In this case if we follow the notation of Arts. 42 and 43, the resolved part of the effective force in the direction of the axis of z is the difference of the values of $\Sigma m\frac{dz}{dt}$ just before and just after the action of the impulses, and this is the same as the difference of the values of $M\frac{d\bar{z}}{dt}$ at the same instants. In the same way the moment of the effective forces about the axis of z will be the difference of the values of

$$\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$$

just before and just after the action of the impulses.

We may therefore extend the general proposition of Art. 45 to impulsive forces in the following manner.

Let (u, v, w), (u', v', w') be the velocities of the centre of gravity of any rigid body of mass M just before and just after the action of the impulses resolved parallel to any three fixed rectangular axes. Let (h_1, h_2, h_3), (h_1', h_2', h_3') be the three moments of the momentum relative to the centre of gravity about three rectangular axes fixed in direction and meeting at the centre of gravity, the moments being taken just before and just after the impulses. Then the effective forces of the body are equivalent to the three effective forces $M(u'-u)$, $M(v'-v)$, $M(w'-w)$ acting at the centre of gravity parallel to the rectangular axes together with the three effective couples $(h_1'-h_1)$, $(h_2'-h_2)$, $(h_3'-h_3)$ about those axes.

These effective forces and couples being reversed will be in equilibrium with the impressed forces. The equations of equilibrium may then be formed according to the rules of statics.

EXAMPLES.

1. Two particles moving in the same plane are projected in parallel but opposite directions with velocities inversely proportional to their masses. Find the motion of their centre of gravity.

2. A person is placed on a perfectly smooth table, show how he may get off.

3. Explain how a person sitting on a chair is able to move the chair across the room by a series of jerks, without touching the ground with his feet.

4. A person is placed at one end of a perfectly rough board which rests on a smooth table. Supposing he walks to the other end of the board, determine how much the board has moved. Supposing that he stepped off the board, show how to determine its subsequent motion.

5. The motion of the centre of gravity of a shell shot from a gun in vacuo is a parabola, and its motion is unaffected by the bursting of the shell.

6. A rod revolving uniformly in a horizontal plane round a pivot at its extremity suddenly snaps in two: determine the motion of each part.

7. A circular disc is capable of motion about a horizontal tangent which rotates with uniform angular velocity Ω about a vertical axis through the point of junction which is fixed. Prove that if the disc be inclined at a constant angle α to the horizon, $\Omega^2 \sin\alpha = \frac{4g}{5a}$.

CHAPTER III.

MOTION ABOUT A FIXED AXIS.

51. *A rigid body can turn freely about an axis fixed in the body and in space, to find the moment of the effective forces about the axis of rotation.*

Let any plane passing through the axis and fixed in space be taken as a plane of reference, and let θ be the angle which any other plane through the axis and fixed in the body makes with the first plane. Let m be the mass of any element of the body, r its distance from the axis, let ϕ be the angle a plane through the axis and the element m makes with the plane of reference.

The velocity of the particle m is $r\dfrac{d\phi}{dt}$ in a direction perpendicular to the plane containing the axis and the particle. The moment of the momentum of this particle about the axis is clearly $mr^2\dfrac{d\phi}{dt}$. Hence the moment of the momenta of all the particles is $\Sigma\left(mr^2\dfrac{d\phi}{dt}\right)$. Since the particles of the body are rigidly connected with each other, it is obvious that $\dfrac{d\phi}{dt}$ is the same for every particle, and equal to $\dfrac{d\theta}{dt}$. Hence the moment of the momenta of all the particles of the body about the axis is $\Sigma mr^2 \frac{d\theta}{dt}$, *i.e.* the moment of inertia of the body about the axis multiplied into the angular velocity.

The accelerations of the particle m are $r\frac{d^2\phi}{dt^2}$ and $-r\left(\frac{d\phi}{dt}\right)^2$ perpendicular to, and along the directions in which r is measured, the moment of the moving forces of m about the axis is $mr^2\frac{d^2\phi}{dt^2}$, hence the moment of the moving forces of all the particles of the body about the axis is $\Sigma\left(mr^2\frac{d^2\phi}{dt^2}\right)$. By the same reasoning as before this is equal to $\Sigma mr^2\frac{d^2\phi}{dt^2}$ *i.e.* the moment of inertia of the body about the axis into the angular acceleration.

52. *To determine the motion of a body about a fixed axis under the action of any forces.*

By D'Alembert's principle the effective forces when reversed will be in equilibrium with the impressed forces. To avoid introducing the unknown reactions at the axis, let us take moments about the axis.

First, let the forces be impulsive. Let ω, ω' be the angular velocities of the body just before and just after the action of the forces. Then, following the notation of the last article,

$$\omega' . \Sigma mr^2 - \omega . \Sigma mr^2 = L,$$

where L is the moment of the impressed forces about the axis;

$$\therefore \omega' - \omega = \frac{\text{moment of forces about axis}}{\text{moment of inertia about axis}}.$$

This equation will determine the change in the angular velocity produced by the action of the forces.

Secondly, let the forces be finite. Then taking moments about the axis, we have

$$\frac{d^2\theta}{dt^2}\cdot\Sigma mr^2 = L,$$

$$\therefore \frac{d^2\theta}{dt^2} = \frac{\text{moment of forces about axis}}{\text{moment of inertia about axis}}.$$

This equation when integrated will give the values of θ and $\frac{d\theta}{dt}$ at any time. Two undetermined constants will make their appearance in the course of the solution. These are to be determined from the given initial values of θ and $\frac{d\theta}{dt}$. Thus the whole motion can be found.

53. *A perfectly rough circular horizontal board is capable of revolving freely round a vertical axis through its centre. A man whose weight is equal to that of the board walks on and round it at the edge: when he has completed the circuit what will be his position in space?*

Let a be the radius of the board, Mk^2 its moment of inertia about the vertical axis. Let ω be the angular velocity of the board, ω' that of the man about the vertical axis at any time. And let F be the action between the feet of the man and the board.

The equation of motion of the board is by Art. 52

$$Mk^2\frac{d\omega}{dt} = -Fa \dots\dots\dots\dots(1).$$

The equation of motion of the man is by Art. 46

$$Ma\frac{d\omega'}{dt} = F \dots\dots\dots\dots(2).$$

Eliminating F and integrating, we get

$$k^2\omega + a^2\omega' = 0,$$

the constant being nothing, because the man and the board start from rest. Let θ, θ' be the angles described by the board and man round the vertical axis. Then $\omega = \frac{d\theta}{dt}$, $\omega' = \frac{d\theta'}{dt}$, and

$$k^2\theta + a^2\theta' = 0.$$

Hence, when $\theta' - \theta = 2\pi$, we have $\theta' = \frac{k^2}{k^2 + a^2} 2\pi$. This gives the angle in space described by the man.

If $k^2 = \frac{a^2}{2}$ we have $\theta' = \frac{2}{3}\pi$. Let V be the mean relative velocity with which the man walks along the board. Then

$$\omega' - \omega = \frac{V}{a};$$

$$\therefore\ \omega = -\frac{Va}{k^2 + a^2} = -\frac{2}{3}\frac{V}{a}.$$

This gives the mean angular velocity of the board.

On the Pendulum.

54. *A body moves about a fixed horizontal axis acted on by gravity only, to determine the motion.*

Take the vertical plane through the axis as the plane of reference, and the plane through the axis and the centre of gravity as the plane fixed in the body. Then the equation of motion is

$$\frac{d^2\theta}{dt^2} = \frac{\text{moment of forces}}{\text{moment of inertia}} \qquad \text{(1)}$$

$$= -\frac{Mgh \sin\theta}{M(k^2 + h^2)},$$

where h is the distance of the centre of gravity from the axis and Mk^2 is the moment of inertia of the body about an axis through the centre of gravity parallel to the fixed axis. Hence

$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2 + h^2}\sin\theta = 0 \qquad \text{(2).}$$

The equation (2) cannot be integrated in finite terms, but if the oscillations be small, we may reject the cubes and higher powers of θ and the equation will become

$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2 + h^2}\theta = 0.$$

Hence the time of a complete oscillation is $2\pi\sqrt{\dfrac{k^2 + h^2}{gh}}$. If h and k be measured in feet and $g = 32{\cdot}18$, this formula gives the time in seconds.

The equation of motion of a particle of any mass suspended by a string l is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}.\sin\theta = 0 \text{......................} (3),$$

which may be deduced from equation (2) by putting $k = 0$ and $h = l$. Hence the angular motions of the string and the body under the same initial conditions will be identical if

$$l = \frac{k^2 + h^2}{h} \text{..............................} (4).$$

This length is called the "length of the simple equivalent pendulum."

Through G, the centre of gravity of the body, draw a perpendicular to the axis of revolution cutting it in C. Then C is called the *centre of suspension.* Produce CG to O so that $CO = l$. Then O is called the *centre of oscillation.* If the whole mass of the body were collected at the centre of oscillation and suspended by a thread to the centre of suspension, its angular motion and time of oscillation would be the same as that of the body under the same initial circumstances.

The equation (4) may be put under another form. Since $CG = h$ and $OG = l - h$, we have

$$CG \,.\, OG = k^2.$$

This equation shows that if O be made the centre of suspension, the axis being parallel to the axis about which k was taken, then C will be the centre of oscillation. Thus *the centres of oscillation and suspension are convertible and the time of oscillation about each is the same.*

If the time of oscillation be given, l is given and the equation (4) will give two values of h. Let these values be h_1, h_2. Let two cylinders be described with that straight line as axis about which the radius of gyration k was taken, and let the radii of these cylinders be h_1, h_2. Then the times of oscillation of the body about any generating lines of these cylinders are the same, and are approximately equal to $2\pi\sqrt{\frac{l}{g}}$.

With the same axis describe a third cylinder whose radius is k. Then $l = 2k + \frac{(h-k)^2}{h}$, hence l is always greater than $2k$, and decreases continually as h decreases and approaches the value k. Thus the length of the equivalent pendulum continually decreases as the axis of suspension approaches from without to the circumference of this third cylinder. When the axis of suspension is a generating line of the cylinder the length of the equivalent pendulum is $2k$. When the axis of suspension is within the cylinder and approaching the centre of gravity the length of the equivalent pendulum continually increases and becomes infinite when the axis passes through the centre of gravity.

The time of oscillation is therefore least when the axis is a generating line of the circular cylinder whose radius is k. But the time about the axis thus found is not an absolute minimum. It is a minimum for all axes drawn parallel to a given straight line in the body. To find the axis about which the time is absolutely a minimum we must find the axis about which k is a minimum. Now it is proved in Art. 16, that of all axes through G the axis about which the moment of inertia is least or greatest is one of the principal axes. Hence the axis about which the time of oscillation is a minimum is parallel to that principal axis through G about which the moment of inertia is least. And if Mk^2 be the moment of inertia about that axis, the axis of suspension is at a distance k measured in any direction from the principal axis.

55. In a clock which is regulated by a pendulum, it is necessary that the time of oscillation should be invariable. As all substances expand and contract with every alteration of temperature, it is clear that the distance of the centre of gravity of the pendulum from the axis and the moment of inertia about that axis will be continually altering. The length of the simple equivalent pendulum does not however depend on either of these elements simply, but on their ratio. If then we can construct a pendulum such that the expansion or contraction of its different parts does not alter this ratio, the time of oscillation will be unaffected by any changes of temperature. For an account of the various methods of accomplishing this which have been suggested, we refer the reader to any treatise* on clocks. We shall here only notice for the sake of illustration one simple construction, which has received general approval.

Mercury is enclosed in a cast iron cylindrical jar into the top of which an iron rod is screwed. This rod is then suspended in the usual manner from a fixed point. The downward expansion of the iron on any increase of temperature tends to lower the centre of oscillation, but the upward expansion of the mercury tends on the contrary to raise it. It is required to determine the condition that the position of the centre of oscillation may on the whole be unaltered.

Let Mk^2 be the moment of inertia of the iron jar and rod about the axis of suspension, c the distance of their common centre of gravity from that axis. Let l be the length of the pendulum from the point of suspension to the bottom of the jar, a the internal radius of the cylindrical jar. Let nM be the mass of the mercury, h the height it occupies in the jar. Then the length L of the simple equivalent pendulum is

$$L = \frac{n\left\{\frac{1}{3}\left(\frac{h}{2}\right)^2 + \frac{a^2}{4} + \left(l - \frac{h}{2}\right)^2\right\} + k^2}{n\left\{l - \frac{h}{2}\right\} + c}$$

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* Reid on Clocks; Denison's treatise on Clocks and Clockmaking in Weale's Series, 1867; Captain Kater's treatise on Mechanics in Lardner's Cyclopædia, 1830.

$$=\frac{n\left(\frac{h^2}{3}-lh+l^2+\frac{a^2}{4}\right)+k^2}{n\left(l-\frac{h}{2}\right)+c}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\text{E}).$$

Let the linear expansion of the substance which forms the rod and jar be denoted by α and that of mercury be β for each degree of the thermometer. If the thermometer used be Fahrenheit's, we have $\alpha = \cdot 0000065668$, $\beta = \cdot 00003336$, according to some experiments of Dulong and Petit. Thus we see that α and β are so small that their squares may be neglected. In calculating the height of the mercury it must be remembered that the jar expands laterally, and thus the relative vertical expansion of the mercury is $3\beta - 2\alpha$, which we shall represent by γ.

If then the temperature of every part be increased t^0, we have a, l, k, c, all increased in the ratio $1+\alpha t : 1$, while h is increased in the ratio $1+\gamma t : 1$. Since L is to be unaltered, we have

$$\left(\frac{dL}{da}a+\frac{dL}{dl}l+\frac{dL}{dk}k+\frac{dL}{dc}c\right)\alpha+\frac{dL}{dh}h\gamma=0.$$

But L is a homogeneous function of one dimension, hence

$$\frac{dL}{da}a+\frac{dL}{dl}l+\frac{dL}{dk}k+\frac{dL}{dc}c+\frac{dL}{dh}h=L.$$

The condition becomes therefore by substitution

$$\frac{\alpha}{\alpha-\gamma}=\frac{h}{L}\,\frac{dL}{dh}.$$

Let A, B be the numerator and denominator of the expression for L given by equation (E). Then taking the logarithmic differential

$$\frac{1}{L}\,\frac{dL}{dh}=\frac{n\left(\frac{2}{3}h-l\right)}{A}+\frac{\frac{1}{2}n}{B}$$

$$=\frac{n}{B}\left(\frac{\frac{2}{3}h-l}{L}+\frac{1}{2}\right).$$

Hence the required condition is

$$\frac{\alpha}{\gamma-\alpha}=\frac{h}{l-\frac{h}{2}+\frac{c}{n}}\cdot\left(\frac{l-\frac{2}{3}h}{L}-\frac{1}{2}\right).$$

56. The oscillations of a rigid body may be made use of to determine the numerical value of the accelerating force of gravity. Let L be the length of a simple equivalent pendulum of any body, and let T be the time of a complete oscillation. Then we have when the oscillations are small,

$$T=2\pi\sqrt{\frac{L}{g}}.$$

Thus g can be determined as soon as L and T are known.

The simplest body to make use of for this purpose is a straight rod, drawn as a wire, and suspended from one extremity. It is easily proved that the centre of oscillation is at a distance from the point of suspension two-thirds of the length of the rod. Thus L is known. T may be found by comparing its oscillations with those of the pendulum of a clock.

By inverting the rod and taking the mean of the results in each position any error arising from want of uniformity in density or figure may be partially obviated. But it is very difficult to obtain a rod so uniform as to give results sufficiently accordant with each other. Captain Kater therefore proposed to use the property (Art. 54) of the convertibility of the centres of suspension and oscillation to obtain more accurate results. *Phil. Trans.* 1818.

Let a body, furnished with a moveable weight, be provided with a point of suspension C, and another point on which it may vibrate, fixed as nearly as can be estimated in the centre of oscillation O, and in a line with the point of suspension and the centre of gravity. The oscillations of the body must now be observed when suspended from C and also when suspended from O. If the vibrations in each position should not be equal in equal times, they may readily be made so by shifting the moveable weight. When this is done, the distance between the two points C and O is the length of the simple equivalent pendulum. Thus the length L and the corresponding time T of vibration will be found uninfluenced by any irregularity of density or figure. In these experiments, after numerous trials of spheres, &c. knife edges were preferred as a means of support. At the centres of suspension and oscillation there were two triangular apertures to admit the knife edges on which the body rested while making its oscillations.

Having thus the means of measuring the length L with accuracy, it remains to determine the time T. This is effected by comparing the vibrations of the body with those of a clock. The time of a single vibration or of any small arbitrary number of vibrations cannot be observed directly, because this would require the fraction of a second of time, as shown by the clock, to be estimated either by the eye or ear. The vibrations of the body may be counted, and here there is no fraction to be estimated, but these vibrations will not probably fit in with the oscillations of the clock pendulum, and the differences must be estimated. This defect is overcome by "the method of coincidences." Supposing the time of vibration of the clock to be a little less than that of the body, the pendulum of the clock will gain on the body, and at length at a certain vibration the two will for an instant coincide. The two pendulums will now be seen to separate and after a time will again approach each other, when the same phenomenon will take place. If the two pendulums continue to vibrate with perfect uniformity, the number of oscillations of the pendulum of the clock in this interval will be an integer, and the number of oscillations of the body in the same interval will be less by one complete oscillation than that of the pendulum of the clock. Hence by a simple proportion the time of a complete oscillation may be found.

The coincidences were determined in the following manner. Certain marks made on the two pendulums were observed by a telescope at the lowest point of their arcs of vibration. The field of view was limited by a diaphragm to a narrow aperture across which the marks were seen to pass. At each succeeding vibration the clock pendulum follows the other more closely, and at last the clock-mark completely covers the other during their passage across the field of view of the telescope. After a few vibrations it appears again preceding the other. The time of disappearance was generally considered as the time of coincidence of the vibrations, though in strictness the mean of the times of disappearance and reappearance ought to have been taken, but the error thus produced is evidently very small. *Encyc. Met.* Figure of the Earth. In the experiments made in Harton coal-pit in 1854, the Astronomer Royal used Kater's method of observing the pendulum, (*Phil. Trans.* 1856).

The value of T thus found will require several corrections. These are called "Reductions." If the centre of oscillation do not describe a cycloid, allowance must be made for the alteration of time depending on the arc described. This is called "the reduction to infinitely small arcs." If the point of support be not absolutely fixed, another correction is required (*Phil. Trans.* 1831). The effect of the buoyancy and the resistance of the air must also be allowed for. This is the "reduction to a vacuum." The length L must also be corrected for changes of temperature.

The time of an oscillation thus corrected enables us to find the value of gravity at the place of observation. A correction is now required to reduce this result to what it would have been at the level of the sea. The attraction of the intervening land must be allowed for by Dr Young's rule (*Phil. Trans.* 1819). We thus obtain the force of gravity at the level of the sea, supposing all the land above this level were cut off and the sea constrained to keep its present level. As the sea would tend in such a case to change its level, further corrections are still necessary if we wish to reduce the result to the surface of that spheroid which most nearly represents the earth. See *Camb. Phil. Trans.* Vol. x.

57. There is another use to which the experimental determination of the length of a simple equivalent pendulum may be applied. "It has been adopted as a standard of length on account of being invariable and capable at any time of recovery. An Act of Parliament, 5 Geo. IV. defines

the yard to contain 36 such parts, of which parts there are 39·1393 in the length of a pendulum vibrating seconds of mean time in the latitude of London in vacuo at the level of the sea at temperature 62° F. The Commissioners, however, appointed to consider the mode of restoring the standards of weight and measure which were lost by fire in 1834, report that several elements of reduction of pendulum experiments are yet doubtful or erroneous, so that the results of a convertible pendulum are not so trustworthy as to serve for supplying a standard of length; and they recommend a material standard, the distance namely between two marks on a certain bar of metal under given circumstances, in preference to any standard derived from measuring phenomena in nature (*Report*, 1841)." Griffin's *Dynamics of a Rigid Body*, page 24.

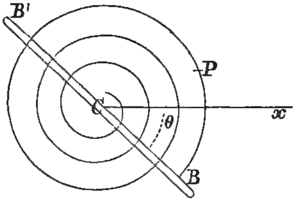
58. A rod $B'CB$ can turn freely about its centre of gravity C which is fixed, and is acted on by a very fine spiral spring AB. The spring has one end A fixed in position in such a manner that the tangent at A is also fixed, and has the other end B attached to the rod so that the tangent at B makes a constant angle with the rod. The rod being turned through any angle, it is required to find the time of oscillation. This is the construction used in watches, just as the pendulum is used in clocks, to regulate the motion.

Let Cx be the position of the rod when in equilibrium, and let θ be the angle the rod makes with Cx at any time t, Mk^2 the moment of inertia of the rod about C. Let ρ be the radius of curvature at any point P of the spring, ρ_0 the value of ρ when in equilibrium. Let (x, y) be the co-ordinates of P referred to C as origin and Cx as axis of x. Let us consider the forces which act on the rod and the portion BP of the spring. The forces on the rod are X, Y the resolved parts of the reaction at C parallel to the axes of co-ordinates, and the reversed effective forces which are equivalent to a couple $Mk^2\dfrac{d^2\theta}{dt^2}$. The forces on the spring are, the reversed effective forces which are so small that they may be neglected, and the resultant action across the section of the spring at P. This resultant action is equivalent to a couple

$E\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)$, and a force acting at P. Taking moments about P we have

$$Mk^2\frac{d^2\theta}{dt^2}=-E\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)-Xy+Yx.$$

This equation is true whatever point P may be chosen. Considering the left side constant at any moment and (x, y) variable, this becomes the intrinsic equation to the form of the spring.



Let $BP=s$, multiply this equation by ds and integrate along the whole length l of the spiral spring, we have

$$Mk^2\frac{d^2\theta}{dt^2}l=-E\int\left(\frac{ds}{\rho}-\frac{ds}{\rho_0}\right)+Y\int xds-X\int yds.$$

Now $\frac{ds}{\rho}$ is the angle between two consecutive normals, hence $\int\frac{ds}{\rho}$ is the angle between the extreme normals. Now at A the normal to the spring is fixed throughout the motion, therefore $\int\left(\frac{ds}{\rho}-\frac{ds}{\rho_0}\right)$ is the angle between the normals at B in the two positions in which $\theta=\theta$ and $\theta=0$. But since the normal at B makes a constant angle with the rod, this angle is the angle θ which the rod makes with its position of equilibrium. Also if $\bar{x}$, $\bar{y}$ be the co-ordinates of the centre of gravity of the spring at the time t, we have $\int xds=\bar{x}l$, $\int yds=\bar{y}l$. Hence the equation of motion becomes

$$Mk^2\frac{d^2\theta}{dt^2}=-\frac{E}{l}\theta+Y\bar{x}-X\bar{y}.$$

Let us suppose that in the position of equilibrium there is no pressure on the axis C, then X and Y will, throughout the motion, be small quantities of the order θ. Let us also suppose that the fulcrum C is placed over the centre of gravity of the spring when at rest. Then if the number of spiral turns of the spring be numerous and if each turn be nearly circular, the centre of gravity will never deviate far from C. So that the terms $Y\bar{x}$ and $X\bar{y}$ are each the product of two small quantities, and are therefore at least of the second order. Neglecting these terms we have

$$Mk^2 \frac{d^2\theta}{dt^2} = -\frac{E}{l}\,\theta.$$

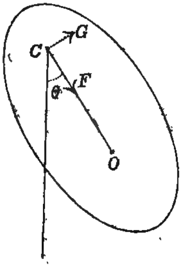
Hence the time of oscillation is $2\pi\sqrt{\dfrac{Mk^2l}{E}}$.

This brief discussion of the motion of a watch balance is taken from a memoir presented to the Academy of Sciences. The reader is referred to an article in Liouville's Journal, 1860, for a further investigation of the conditions necessary for isochronism and for a determination of the best forms for the spring.

Pressures on the fixed axis.

59. *A body moves about a fixed axis under the action of any forces, to find the pressures on the axis.*

First, Suppose the body and the forces to be symmetrical about the plane through the centre of gravity perpendicular to the axis. Then it is evident that the pressures



on the axis are reducible to a single force at C the centre of suspension.

Let F, G be the actions of the point of support on the body resolved along and perpendicular to CO, where O is the centre of gravity. Let X, Y be the sum of the resolved parts of the impressed forces in the same directions, and L their moment round C.

Let $CO = h$ and $\theta =$ angle which CO makes with any straight line fixed in space.

Taking moments about C, we have

$$\frac{d^2\theta}{dt^2} = \frac{L}{M(k^2+h^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The motion of the centre of gravity is the same as if all the forces acted at that point. Now it describes a circle round C; hence, taking the tangential and normal resolutions, we have

$$h\frac{d^2\theta}{dt^2} = \frac{Y+G}{M} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$-h\left(\frac{d\theta}{dt}\right)^2 = \frac{X+F}{M} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Equation (1) gives the values of $\frac{d^2\theta}{dt^2}$ and $\frac{d\theta}{dt}$. and then the pressures may be found by equations (2) and (3).

If the only force acting on the body be that of gravity, and if the body start from rest in that position which makes CO horizontal, then we have

$$X = Mg\cos\theta, \quad Y = -Mg\sin\theta, \quad L = -Mgh\sin\theta;$$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{gh}{k^2+h^2}\sin\theta,$$

integrating, we have

$$\left(\frac{d\theta}{dt}\right)^2 = C + \frac{2gh}{k^2+h^2}\cos\theta,$$

but when $\theta = \frac{\pi}{2}$, $\frac{d\theta}{dt}$ vanishes, therefore $C = 0$; substituting these values (2) and (3), we get

$$\left.\begin{aligned} -F &= Mg\cos\theta \, . \, \frac{k^2 + 3h^2}{k^2 + h^2} \\ G &= Mg\sin\theta \, . \, \frac{k^2}{k^2 + h^2} \end{aligned}\right\},$$

where θ is the angle which CO makes with the vertical.

Let ψ be the angle the direction of the pressure at C makes with the line CO, the angle being measured from CO downwards to the left, then

$$\cot\psi = \left(1 + 3\frac{h^2}{k^2}\right)\cot\theta,$$

which is a convenient formula to determine the direction of the pressure*.

60. *Secondly.* Suppose either the body or the forces not to be symmetrical.

Let the fixed axis be taken as the axis of z with any origin and plane of xz. These we shall afterwards so choose as to simplify our process as much as possible. Let $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity at the time t.

Let ω be the angular velocity of the body, f the angular acceleration, so that $f = \frac{d\omega}{dt}$.

---

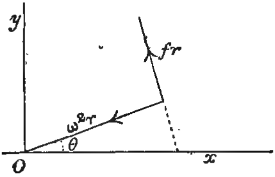
[*] Let mR be the resultant of F and G, and let

$$a = g\,\frac{k^2 + 3h^2}{k^2 + h^2} \text{ and } b = g\,\frac{k^2}{k^2 + h^2},$$

then
$$\frac{\cos^2\psi}{a^2} + \frac{\sin^2\psi}{b^2} = \frac{1}{R^2}.$$

Construct an ellipse with C for centre and axes equal to a and b measured along and perpendicular to CO. Then the resultant pressure varies as the diameter along which it acts. And the direction may be found thus; let the auxiliary circle cut the vertical in V, and let the perpendicular from V on CO cut the ellipse in R. Then CR is the direction of the pressure.

Now every element m of the body describes a circle about the axis, hence its accelerations along and perpendicular to the radius vector r from the axis are $-\omega^2 r$ and fr. Let θ be the angle which r makes with the plane of xz at any time, then from the resolution of forces it is clear that



$$\frac{d^2x}{dt^2} = -\omega^2 r \cos\theta - fr\sin\theta$$

$$\left.\begin{aligned} &= -\omega^2 x - fy \\ \frac{d^2y}{dt^2} &= -\omega^2 y + fx \end{aligned}\right\}.$$

These equations may also be obtained by differentiating the equations $x = r\cos\theta$, $y = r\sin\theta$ twice, remembering that r is constant.

Conceive the body to be fixed to the axis at two points, distant a and a' from the origin, and let the reactions of the points on the body resolved parallel to the axes be respectively F, G, H; F', G', H'.

The equations of motion then give

$$\begin{aligned} \Sigma mX + F + F' &= \Sigma m\frac{d^2x}{dt^2} \\ &= \Sigma m(-\omega^2 x - fy) \\ &= -\omega^2 M\bar{x} - fM\bar{y} \quad \ldots\ldots\ldots\ldots\ldots (1), \end{aligned}$$

$$\begin{aligned} \Sigma mY + G + G' &= \Sigma m\frac{d^2y}{dt^2} \\ &= \Sigma m(-\omega^2 y + fx) \\ &= -\omega^2 M\bar{y} + fM\bar{x} \quad \ldots\ldots\ldots\ldots\ldots (2), \end{aligned}$$

$$\Sigma mZ + H + H' = \Sigma m \frac{d^2z}{dt^2}$$
$$= 0 \text{ .... } (3).$$

Taking moments about the axes, we have

$$\Sigma m(yZ - zY) - Ga - G'a' = \Sigma m \left(y \frac{d^2z}{dt^2} - z \frac{d^2y}{dt^2}\right)$$
$$= -\Sigma m \left(z \frac{d^2y}{dt^2}\right)$$
$$= \omega^2 \Sigma myz - f \Sigma mxz \text{ .... } (4),$$

by merely introducing z into the results in (2),

$$\Sigma m(zX - xZ) + Fa + F'a' = \Sigma m \left(z \frac{d^2x}{dt^2} - x \frac{d^2z}{dt^2}\right)$$
$$= -\omega^2 \Sigma mxz - f \Sigma myz \text{ ...... } (5),$$

$$\Sigma m(xY - yX) = \Sigma m \left(x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2}\right)$$
$$= \Sigma mr^2 . \frac{d\omega}{dt}$$
$$= Mk'^2 . f \text{ .... } (6).$$

Equation (6) serves to determine f and ω, and equations (1), (2), (4), (5) then determine F, G, F', G'; H and H' are indeterminate, but their sum is given by equation (3).

Looking at these equations, we see that they would be greatly simplified in two cases.

First, if the axis of z be a principal axis at the origin,

$$\Sigma mxz = 0, \quad \Sigma myz = 0,$$

and the calculation of the right-hand sides of equations (4) and (5) would only be so much superfluous labour. Hence, in attempting a problem of this kind, we should, when possible, so choose the origin that the axis of revolution is a principal axis of the body at that point.

Secondly, except the determination of f and ω by integrating equation (6), the whole process is merely an algebraic substitution of f and ω in the remaining equations. Hence our results will still be correct if we choose the plane of xz to contain the centre of gravity at the moment under consideration; this will make $\bar{y}=0$, and thus equations (1) and (2) will be simplified.

61. If the forces which act on the body be impulsive, the equations will require some alterations.

Let ω, ω' be the angular velocities of the body just before and just after the action of the impulses. In the case in which the body and forces are symmetrical, the equations (1), (2), (3) of Art. 59 become respectively

$$\omega' - \omega = \frac{L}{M(k^2+h^2)} \qquad (1),$$

$$h(\omega' - \omega) = \frac{Y+G}{M} \qquad (2),$$

$$0 = \frac{X+F}{M} \qquad (3),$$

where all the letters have the same meaning as before, except that F, G, X, Y are now impulsive instead of finite forces.

Let us next consider the case in which the forces on the body are not symmetrical. Let u, v, w, u', v', w' be the velocities resolved parallel to the axes of any element m whose co-ordinates are x, y, z. Then $u=-y\omega$, $u'=-y\omega'$, $v=x\omega$, $v'=x\omega'$, and w, w' are both zero.

The several equations of Art. 60 will then be replaced by the following:

$$\begin{aligned}\Sigma X + F + F' &= \Sigma m(u'-u)\\ &= -\Sigma my(\omega'-\omega)\\ &= -M\bar{y}(\omega'-\omega) \qquad (1).\end{aligned}$$

$$\begin{aligned}\Sigma Y + G + G' &= \Sigma m(v'-v)\\ &= \Sigma mx(\omega'-\omega)\\ &= M\bar{x}(\omega'-\omega) \qquad (2),\end{aligned}$$

$$\Sigma Z + H + H' = 0 \qquad (3),$$

$$\begin{aligned}\Sigma\,(yZ - zY) - Ga - G'a' &= \Sigma m\,\{y\,(w' - w) - z\,(v' - v)\}\\ &= -\Sigma mxz\,.\,(\omega' - \omega)\,\ldots\ldots\ldots(4),\\ \Sigma\,(zX - xZ) + Fa + F'a' &= \Sigma m\,\{z\,(u' - u) - x\,(w' - w)\}\\ &= -\Sigma myz\,.\,(\omega' - \omega)\,\ldots\ldots\ldots(5),\\ \Sigma\,(xY - yX) \qquad\qquad &= \Sigma m\,(x^2 + y^2)\,.\,(\omega' - \omega)\,\ldots\,(6).\end{aligned}$$

These six equations are sufficient to determine ω', F, F', G, G' and the sum $H + H'$ of the two pressures along the axis.

These equations admit of simplification when the origin can be so chosen that the axis of rotation is a principal axis at that point. In this case the right-hand sides of equations (4) and (5) vanish. Also if the plane of xz be chosen to pass through the centre of gravity of the body, we have $\bar{y} = 0$, and the right-hand side of equation (1) vanishes.

62. It is sometimes important to determine beforehand whether the pressures on the fixed axis can be reduced to a single resultant or not. The general condition is somewhat complicated, but there are some cases which are very simple.

Let the axis of revolution be a principal axis at some point O in its length, and let the forces be such that their moments about any two axes at O at right angles to the axis of revolution are zero. In this case, if we take O for origin, the terms $\Sigma m\,(yZ - zY)$ and $\Sigma m\,(zX - xZ)$ on the left-hand sides of the equations (4) and (5) are zero. The right-hand sides also vanish, because the axis of z is a principal axis at the origin. Hence the equations become

$$\left.\begin{aligned}Ga + G'a' &= 0\\ Fa + F'a' &= 0\end{aligned}\right\}.$$

These show that the moments of the pressures on the fixed points of the axis about the co-ordinate axes are zero. Hence the pressures are together equivalent to a single force acting at O.

The magnitude of this single pressure may be found very conveniently by the method of Art. 59, even though there be no symmetry either of the body or the forces.

For example, suppose the body to be in motion about a fixed horizontal axis, parallel to a principal axis at the centre of gravity G. Let GO be a perpendicular to the axis meeting it in O. Then by Art. 29 the axis is a principal axis at O. Also the weight of the body acting at G has no moment about any axis through O perpendicular to the axis of rotation. Hence by what precedes, the pressures on the axis are equivalent to a single force at O.

63. If a body be set in rotation about any axis which is a principal axis at some point O in its length, and if there be no impressed forces acting on the body, it follows at once from these conditions that the pressures on the axis are equivalent to a single resultant force acting at O. Hence if O be fixed in space, the body will continue to rotate about that axis as if it also were fixed in space. Such an axis is called a *permanent axis of rotation* at the point O.

If the body be entirely free and yet turning about an axis of rotation which does not alter its position in space, we may suppose any point we please in the axis to be fixed. In this case the axis must be a principal axis at every point of its length. It must therefore by Art. 24 pass through the centre of gravity.

The existence of principal axes was first established by Segner in the work *Specimen Theoriæ Turbinum*. His course of investigation is the opposite of that pursued in this treatise. He defines a principal axis to be such that when a body revolves round it the forces arising from the rotation have no tendency to alter the position of the axis. From this dynamical definition he deduces the geometrical properties of these axes. Cayley's report to the British Association on the special problems of Dynamics, 1862. Bossut, *Histoire de Mathematique*, Tome II.

64. There is another case in which the pressures on the fixed axis may be reduced to a single resultant pressure. Supposing such a single resultant pressure to exist, we can take as origin that point of the axis at which it is intersected by the single resultant. Then the moments of the two pressures on the axis of rotation about the co-ordinate axes

will vanish. Hence the equations (4), (5), and (6) of Art. 61 become

$$\left.\begin{aligned} L &= \omega^2\Sigma myz - f\Sigma mxz \\ M &= -\omega^2\Sigma mxz - f\Sigma myz \\ N &= Mk'^2 f \end{aligned}\right\},$$

where we have written L, M, N for the three moments $\Sigma m\,(yZ - zY)$, &c. of the impressed forces about the co-ordinate axes.

If the body start from rest we have at that instant $\omega = 0$.

The plane of the couple whose resolved parts about the axes are L, M, N is known by Statics to be

$$LX + MY + NZ = 0,$$

or in our case,

$$-\Sigma mxzX - \Sigma myzY + Mk'^2Z = 0.$$

Let the momental ellipsoid at the fixed point be constructed, and let its equation be

$$AX^2 + BY^2 + CZ^2 - 2DYZ - 2EZX - 2FXY = \epsilon^4.$$

The equation to the diametral plane of the axis of Z is

$$-EX - DY + CZ = 0.$$

Hence the plane of the resultant couple must be the diametral plane of the axis of revolution.

Since the pressures on the axis are equivalent to a single resultant force acting at some point O of the axis, we may suppose this point alone to be fixed and the axis of rotation to be otherwise free. If then a body at rest with one point fixed be acted on by any couple, it will begin to rotate about the diametral line of the plane of the couple with regard to the momental ellipsoid at the fixed point.

Thus the body will begin to rotate about a perpendicular to the plane of the couple only when the plane of the couple is parallel to a principal plane of the body at the fixed point.

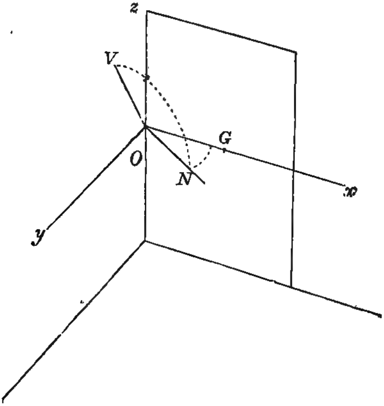
If the acting couple be an impulsive couple, the equations of motion, by Art. 61, will be the same as those obtained above when ω is put zero and ω' written for f. Hence the same conclusion will follow.

The body will not in general continue to rotate about the diametral line.

65. EXAMPLE. *A door is suspended by two hinges from a fixed axis making an angle α with the vertical. Find the motion and pressures on the hinges.*

Since the fixed axis is evidently a principal axis at the middle point, we shall take this point for origin. Also we shall take the plane of xz so that it contains the centre of gravity of the door at the moment under consideration.

The only force acting on the door is gravity, which may be supposed to act at the centre of gravity. We must first resolve this parallel to the axes. Let ϕ be the angle the plane of the door makes with a vertical plane through the axis of suspension. If we draw a plane ZON such that its trace ON on the plane of XOY makes an angle ϕ with the axis of x, this will be the vertical plane through the axis; and if we draw OV in this plane making $ZOV = \alpha$, OV will be vertical. Hence the resolved parts of gravity are



$$X = g \sin\alpha \cos\phi, \quad Y = g \sin\alpha \sin\phi,$$

$$Z = -g \cos\alpha.$$

The six equations of motion are

$$Mg\sin\alpha\cos\phi+F+F'=\Sigma m\frac{d^2x}{dt^2}$$
$$=\Sigma m\,.\,(-\omega^2x)$$
$$=-\omega^2M\bar{x}\;\ldots\ldots\ldots\ldots\ldots\;(1),$$

$$Mg\sin\alpha\sin\phi+G+G'=\Sigma m\frac{d^2y}{dt^2}$$
$$=\Sigma m\,(fx)$$
$$=fM\bar{x}\;\ldots\ldots\ldots\ldots\ldots\ldots\;(2),$$

$$-Mg\cos\alpha+H+H'=\Sigma m\frac{d^2z}{dt^2}$$
$$=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(3),$$

$$-Ga+G'a=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(4),$$

$$Mg\cos\alpha\bar{x}+Fa-F'a=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(5),$$

because the fixed axis is a principal axis

$$-Mg\sin\alpha\sin\phi\,.\,\bar{x}=Mk'^2\,.\,\frac{d^2\phi}{dt^2}\;\ldots\ldots\ldots\ldots\;(6).$$

Integrating the last equation, we have

$$C+2g\sin\alpha\cos\phi\bar{x}=k'^2\omega^2.$$

Suppose the door to be initially placed at rest, with its plane making an angle β with the vertical plane through the axis; then when $\phi=\beta$, $\omega=0$; hence

$$\left.\begin{aligned}&k'^2\omega^2=2g\bar{x}\sin\alpha\,(\cos\phi-\cos\beta)\\ \text{and}\quad&k'^2f=-g\sin\alpha\sin\phi\,.\,\bar{x}\end{aligned}\right\}.$$

By substitution in the first four equations F, F', G, G', may be found.

The Centre of Percussion.

66. When the fixed axis is given and the body can be so struck that there is no impulsive pressure on the axis, any point in the line of action of the force is called a *centre of percussion.*

When the line of action of the blow is given, the axis about which the body begins to turn is called the *axis of spontaneous rotation.* It obviously coincides with the position of the fixed axis in the first case.

67. Prop. *A body is capable of turning freely about a fixed axis. To determine the conditions that there shall be a centre of percussion and to find its position.*

Take the fixed axis as the axis of z, and let the plane of xz pass through the centre of gravity of the body. Let X, Y, Z be the resolved parts of the impulse, and let ξ, η, ζ be the co-ordinates of any point in its line of action. Let Mk'^2 be the moment of inertia of the body about the fixed axis. Then since $\bar{y}=0$, the equations of motion are, by Art. 61,

$$\left.\begin{aligned} X &= 0 \\ Y &= M\bar{x}\,(\omega'-\omega) \\ Z &= 0 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (1),$$

$$\left.\begin{aligned} \eta Z - \zeta Y &= -(\omega'-\omega)\,\Sigma mxz \\ \zeta X - \xi Z &= -(\omega'-\omega)\,\Sigma myz \\ \xi Y - \eta X &= (\omega'-\omega)\,.\,Mk'^2 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (2).$$

The impulsive pressures on the fixed axes are omitted because by hypothesis they do not exist.

From these equations we may deduce the following conditions.

I. From (1) we see that $X=0$, $Z=0$, and therefore the force must act perpendicular to the plane containing the axis and the centre of gravity.

II. Substituting from (1) in the first two equations of (2) we have $\Sigma myz = 0$ and $\zeta = \frac{\Sigma mxz}{M\bar{x}}$. Since the origin may be taken anywhere in the axis of rotation, let it be so chosen that $\Sigma mxz = 0$. Then the axis of z must be a principal axis at the point where a plane passing through the line of action of the blow perpendicular to the axis cuts the axis. So that there can be no centre of percussion unless the axis be a principal axis at some point in its length.

III. Substituting from (1) in the last equation of (2) we have $\xi = \frac{k'^2}{h}$. By Art. 54 this is the equation to determine the centre of oscillation of the body about the fixed axis treated as an axis of suspension. Hence the perpendicular distance between the line of action of the impulse and the fixed axis must be equal to the distance of the centre of oscillation from the axis.

If the fixed axis be parallel to a principal axis at the centre of gravity, the line of action of the blow will pass through the centre of oscillation.

The Ballistic Pendulum.

68. A rifle is attached in a horizontal position to a large block of wood which can turn freely about a horizontal axis. The rifle being fired, the recoil causes the pendulum to turn round its axis, until brought to rest by the action of gravity. A piece of tape is attached to the pendulum, and is drawn out of a reel during the backward motion of the pendulum, and thus serves to measure the amount of the angle of recoil. The initial velocity of the bullet is so much greater than that of the pendulum that we may suppose the ball to have left the rifle before the pendulum has sensibly moved from its initial position. The initial momentum of the bullet may be taken as a measure of the impulse communicated to the pendulum.

Let h be the distance of the centre of gravity from the axis of suspension; f the distance from the axis of the rifle to the axis of suspension; c the distance from the axis of suspension to the point of attachment of the tape, m the mass of the bullet; M that of the pendulum and rifle, and n the ratio of M to m; b the chord of the arc of the recoil which is measured by the tape. Let κ' be the radius of gyration of the rifle and pendulum about the axis of suspension, v the initial velocity of the bullet.

The explosion of the gunpowder generates an equal impulsive action on the bullet and on the rifle. Since the initial velocity of the bullet is v, this action is measured by mv. The initial angular velocity generated in the pendulum by this impulse is by Art. 52 $\omega = \frac{mvf}{M\kappa'^2}$. The subsequent motion is given (Art. 54) by the equation

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{\kappa'^2}\sin\theta;$$

$$\therefore \left(\frac{d\theta}{dt}\right)^2 = C + \frac{2gh}{\kappa'^2}\cos\theta,$$

when $\theta = 0$ we have $\frac{d\theta}{dt} = \omega$, and if α be the angle of recoil, when $\theta = \alpha$, $\frac{d\theta}{dt} = 0$.

Hence $$\omega^2 = \frac{2gh}{\kappa'^2}(1 - \cos\alpha).$$

Eliminating ω we have

$$v = \frac{n\kappa'}{f} \cdot 2\sin\frac{\alpha}{2}\sqrt{gh}.$$

But the chord of the arc of the recoil is $b = 2c\sin\frac{\alpha}{2}$;

$$\therefore v = \frac{nb\kappa'}{cf}\sqrt{gh}.$$

The magnitude of κ' may be found experimentally by observing the time of a small oscillation of the pendulum and rifle. If T be a half-time we have $T = \pi\sqrt{\frac{\kappa'^2}{gh}}$, (Art. 54).

The reader will find in the *Phil. Mag.* for June 1854, an account of some recent experiments conducted by S. Haughton on the velocities of rifle bullets. See also Poisson's *Mécanique*, Vol. II. p. 119.

This instrument is called the Ballistic Pendulum, and was invented by Robins.

EXAMPLES.

1. Find the time of the small oscillations of a cube (1) when one side is fixed, (2) when the diagonal of one of its faces is fixed; the axis in both cases being horizontal.

Result. If $2a$ be a side of the cube, the length of the simple equivalent pendulum is in the first case $\frac{4\sqrt{2}}{3}a$, and in the second $\frac{5}{3}a$.

2. An elliptic lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation, prove that the eccentricity is $=\frac{1}{2}$.

3. A circular arc oscillates about an axis through its middle point perpendicular to the plane of the arc. Prove that the length of the simple equivalent pendulum is independent of the length of the arc, and is equal to twice the radius.

4. The density of a rod varies as the distance from one end, find the axis perpendicular to it about which the time of oscillation is a minimum.

Result. The axis passes through either of two points whose distance from the centre of gravity is $\frac{\sqrt{2}}{6}a$, where a is the length of the rod.

5. Find what axis in the area of an ellipse must be fixed that the time of a small oscillation may be a minimum.

Result. The axis must be parallel to the major axis, and bisect the semi-minor axis.

6. A uniform stick hangs freely by one end, the other end being close to the ground. An angular velocity in a vertical plane is then communicated to the stick, and when it has risen through an angle of 90°, the end by which it was hanging is loosed. What must be the initial angular velocity so that on falling to the ground it may pitch in an upright position?

Result. The required angular velocity ω is given by

$$\omega^2 = \frac{3g}{2a} + \frac{g}{2a}\,\frac{\left\{(2n+1)\frac{\pi}{2}\right\}^2}{(2n+1)\frac{\pi}{2}+1},$$

where n is any integer, and $2a$ is the length of the rod.

7. Two bodies can move freely and independently under the action of gravity about the same horizontal axis; their masses are m, m', and the distances of their centres of gravity from the axis are h, h'. If the lengths of their simple equivalent pendulums be L, L', prove that when fastened together the length of the equivalent pendulum will be $\dfrac{mhL + m'h'L'}{mh + m'h'}$.

8. When it is required to regulate a clock without stopping the pendulum, it is usual to add or subtract some small weight from a platform attached to the pendulum. Show that in order to make a given alteration in the going of the clock by the addition of the least possible weight, the platform must be placed at a distance from the point of suspension half the simple equivalent pendulum. Show also that a slight error in the position of the platform will not affect the weight required to be added.

9. A uniform heavy lamina in the form of a sector of a circle is suspended by a horizontal axis parallel to the radius which bisects the arc, and oscillates under the action of

gravity. Show that the pressures on the axis are equivalent to a single force, and find its magnitude. See Art. 62.

10. A solid ellipsoid is fixed at its centre, and is acted on by a couple in a plane whose direction-cosines referred to the principal diameters are (lmn). Prove that the direction-cosines of the initial axis of rotation are proportional to $\frac{l}{b^2+c^2}$, $\frac{m}{c^2+a^2}$ and $\frac{n}{a^2+b^2}$. See Art. 64.

11. Any plane section being taken of the momental ellipsoid of a body at a fixed point, the body may be made to rotate about either of the principal diameters of this section by the application of a couple of the proper magnitude whose axis is the other principal diameter.

For assume the body to be turning uniformly about the axis of z. Then the couples which must act on the body to produce this motion are $L=\omega^2\Sigma myz$, $M=-\omega^2\Sigma mxz$, $N=0$. Then by taking the axis of x such that $\Sigma mxz=0$ we see that the axis of the couple must be the axis of x and the magnitude of the couple will be $L=\omega^2\Sigma myz$.

12. A body having one point O fixed in space is made to rotate about any proposed straight line by the application of the proper couple. The position of the axis of rotation when the magnitude of the couple is a maximum, has been called an axis of maximum reluctance. Show that there are six axes of maximum reluctance, two in each principal plane, each two bisecting the angles between the principal axes in the plane in which they are.

Let the axes of reference be the principal axes of the body at the fixed point, let (l, m, n) be the direction-cosines of the axis of rotation, (λ, μ, ν) those of the axis of the couple G. Then by the last question and the fifth and sixth propositions of Art. 22, we have

$$\frac{\lambda}{(B-C)\,mn}=\frac{\mu}{(C-A)\,nl}=\frac{\nu}{(A-B)\,lm},$$

$$G^2=(A-B)^2l^2m^2+(B-C)^2m^2n^2+(C-A)^2n^2l^2.$$

We have then to make G a maximum by variation of (lmn) subject to the condition $l^2+m^2+n^2=1$. The positions of these axes were first investigated by Mr Walton in the Quarterly Journal of Mathematics, 1865.

13. If a body revolve with uniform angular velocity ω about a fixed axis whose direction-cosines relative to the

principal axes at any point O in its length are (lmn), then the whole effective forces of the body are together equivalent to a linear effective force $M\bar{r}\omega^2$ at O, and a couple whose resolved parts about the axes are $-(B-C)\,mn\omega^2$, $-(C-A)\,nl\omega^2$ and $-(A-B)\,lm\omega^2$, where $\bar{r}$ is the distance of the centre of gravity from the axis.

14. A circular table centre O is supported by three legs AA', BB', CC' which rest on a perfectly rough horizontal floor, and a heavy particle P is placed on the table. Suddenly one leg CC' gives way, show that the table and the particle will immediately separate if pc be greater than κ^2; where p and c are the distances of P and O from the line AB joining the tops of the legs and κ is the radius of gyration of the table and legs about the line $A'B'$ joining the points where the legs rest on the floor.

The condition of separation is that the initial normal acceleration of the point of the table at P should be greater than the normal acceleration of the particle itself.

15. A body at rest with one point fixed in space is acted on by an impulsive couple whose moment is G, the plane of which cuts the momental ellipsoid at the fixed point in a section whose area is Γ. If $M\kappa^2$ be the moment of inertia about the initial axis of rotation and ω the initial angular velocity, prove that $\dfrac{G}{\kappa\omega\Gamma}$ is always the same for the same body, and is equal to $\dfrac{1}{\pi\epsilon^4}\sqrt{\dfrac{ABC}{M}}$.

16. A string without weight is placed round a fixed ellipse whose plane is vertical, and the two ends are fastened together. The length of the string is greater than the perimeter of the ellipse. A heavy particle can slide freely on the string and performs small oscillations under the action of gravity. Prove that the simple equivalent pendulum is the radius of curvature of the confocal ellipse passing through the position of equilibrium of the particle.

CHAPTER IV.

MOTION IN TWO DIMENSIONS.

69. THE position of a body in space of two dimensions may be determined by the co-ordinates of its centre of gravity, and the angle some straight line fixed in the body makes with some straight line fixed in space. These three are called the "co-ordinates of the body," and it is our object to determine them in terms of the time.

The manner in which this may be done has been explained in the second chapter. All that is now necessary is to find an expression for the moment of the momentum

$$h = \Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right)$$

in terms of the co-ordinates of the body. Let θ be the "angular co-ordinate" of the body, *i.e.* the angle some straight line fixed in the body makes with some fixed straight line in space. Let (r', ϕ') be the polar co-ordinates of any particle m referred to the centre of gravity of the body as origin. Then r' is constant throughout the motion, and $\frac{d\phi'}{dt}$ is the same for every particle of the body and equal to $\frac{d\theta}{dt}$. Then the angular momentum

$$\begin{aligned} h = \Sigma m \left(x' \frac{dy}{dt} - y' \frac{dx'}{dt} \right) &= \Sigma m \left(r'^2 \frac{d\phi'}{dt} \right) \\ &= (\Sigma m r'^2) \frac{d\phi'}{dt} \\ &= Mk^2 \frac{d\theta}{dt}, \end{aligned}$$

where Mk^2 is the moment of inertia of the body about its centre of gravity.

The general method of proceeding according to the Article already quoted will therefore be as follows.

Let (x, y) be the co-ordinates of the centre of gravity of any body of the system referred to rectangular axes fixed in space, M the mass of the body. Then the effective forces of the body are together equivalent to two forces measured by $M\frac{d^2x}{dt^2}$, $M\frac{d^2y}{dt^2}$ acting at the centre of gravity parallel to the axes of co-ordinates, together with a couple measured by $Mk^2\frac{d^2\theta}{dt^2}$ tending to turn the body about its centre of gravity in the direction in which θ is measured. By D'Alembert's principle the effective forces of all the bodies, if reversed, will be in equilibrium with the impressed forces. The dynamical equations may then be formed according to the ordinary rules of Statics.

For example, if we took moments about a point whose co-ordinates are (p, q) we should have an equation of the form

$$M\left\{(x-p)\frac{d^2y}{dt^2}-(y-q)\frac{d^2x}{dt^2}\right\}+Mk^2\frac{d^2\theta}{dt^2}=L,$$

where L is the moment of the impressed forces. In this equation (p, q) may be the co-ordinates of any point whatever, whether fixed or moving. Just as in a statical problem, the solution of the equations may frequently be much simplified by a proper choice of the point about which to take moments. Thus if we wished to avoid the introduction into our equations of some unknown reaction, we might take moments about the point of application. So again in resolving our forces we might replace the Cartesian expressions $M\frac{d^2x}{dt^2}$, $M\frac{d^2y}{dt^2}$ by the polar forms

$$M\left\{\frac{d^2r}{dt}-r\left(\frac{d\phi}{dt}\right)^2\right\} \text{ and } M\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right)$$

for the resolved parts parallel and perpendicular to the radius vector. If v be the velocity of the centre of gravity, ρ the radius of curvature of its path, we may sometimes also use with advantage the forms $\frac{dv}{dt}$ and $\frac{v^2}{\rho}$ for the resolved parts of the effective forces along the tangent and radius of curvature of the path of the centre of gravity.

70. Suppose we form the equations of motion of each body by resolving parallel to the axes of co-ordinates and by taking moments about the centre of gravity. We shall get three equations for each body of the form

$$\left.\begin{aligned} M\frac{d^2x}{dt^2} &= F\cos\phi + \dots \\ M\frac{d^2y}{dt^2} &= F\sin\phi + \dots \\ Mk^2\frac{d^2\theta}{dt^2} &= \quad Fp \quad + \dots \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (1),$$

where F is any one of the forces acting on the body, whose resolved parts are $F\cos\phi$, $F\sin\phi$, and whose moment about the centre of gravity is Fp. These we shall call the "Dynamical equations" of the body.

Besides these there will be certain geometrical equations expressing the connections of the system. As every such forced connection is accompanied by a reaction and every reaction by some forced connection, the number of geometrical equations will be the same as the number of unknown reactions in the system.

Having obtained the proper number of equations of motion we proceed to their solution. Two general methods have been proposed.

First Method. Differentiate the geometrical equations twice with respect to t, and substitute for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2\theta}{dt^2}$, from the dynamical equations. We shall then have a sufficient number of equations to determine the reactions. This method will be of great advantage whenever the geometrical equations are of the form

$$Ax + By + C\theta = D \dots\dots\dots\dots\dots\dots (2),$$

where A, B, C, D are constants. Suppose also that the dynamical equations are such that when written in the form (1) they contain only the reactions and constants on the right-hand side without any x, y, or θ. Then, when we substitute in the equation

$$A\frac{d^2x}{dt^2}+B\frac{d^2y}{dt^2}+C\frac{d^2\theta}{dt^2}=0,$$

obtained by differentiating (1), we have an equation containing only the reactions and constants. This being true for all the geometrical relations, it is evident that all the reactions will be constant throughout the motion and their values may be found. Hence when these values are substituted in the dynamical equations (1), their right-hand members will all be constants and the values of x, y, and θ may be found by an easy integration.

If however the geometrical equations are not of the form (2), this method of solution will usually fail. For suppose any geometrical equation took the form

$$x^2+y^2=c^2,$$

containing *squares* instead of *first* powers, then its second differential equation will be

$$x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=0;$$

and though we can substitute for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, we cannot, in general, eliminate the terms $\left(\frac{dx}{dt}\right)^2$ and $\left(\frac{dy}{dt}\right)^2$.

71. *Second Method.* Suppose the system to consist of only one body of mass m, and let the equations of motion be written in the form

$$\left.\begin{aligned} m\frac{d^2x}{dt^2}&=AR+\dots\\ m\frac{d^2y}{dt^2}&=BR+\dots\\ mk^2\frac{d^2\theta}{dt^2}&=CR+\dots\end{aligned}\right\}\dots\dots\dots\dots\dots\dots(2),$$

where A, B, C are the coefficients of some unknown reaction R which may enter into all the equations. Multiplying these equations respectively by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$ and $2\frac{d\theta}{dt}$, and adding, we get

$$2m\frac{dx}{dt}\frac{d^2x}{dt^2}+2m\frac{dy}{dt}\frac{d^2y}{dt^2}+2mk^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}$$

$$=2\left(A\frac{dx}{dt}+B\frac{dy}{dt}+C\frac{d\theta}{dt}\right)R+\dots$$

then it will be proved in a subsequent chapter that, by virtue of the geometrical equations, the coefficient of R will vanish. And in the same way all the other unknown reactions will disappear from the equation.

Integrating this equation we get

$$m\left(\frac{dx}{dt}\right)^2+m\left(\frac{dy}{dt}\right)^2+mk^2\left(\frac{d\theta}{dt}\right)^2 = \text{a known function of } x,\ y,$$
and θ.

If there be two geometrical equations we shall be able to express x and y in terms of θ, and substituting we shall have

$$\frac{d\theta}{dt}=\text{a known function of } \theta.$$

This when solved will enable us to determine x, y, and θ as functions of t.

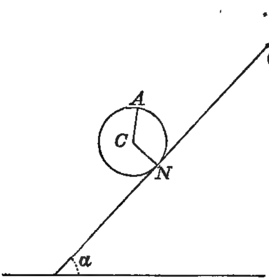
If there be only one geometrical equation there will be only one unknown reaction in the original equations (1). This must be eliminated from the equations by some process different from that described above, and adapted to the particular case in question. It is obvious there cannot be more than two independent geometrical relations, for then no motion would in general be possible.

If there be several bodies in the proposed Dynamical System, the same process will apply. Each set of equations must be multiplied by the factors above described and all the sets must be added together.

This method, with a few exceptions, will give a first integral of the original equations free from any unknown reactions. If the whole system of bodies be so connected by its geometrical relations that only one independent motion of the whole system is possible this one equation will be sufficient to give that motion.

This is called the method of *vis viva*, and the cases of exception will be considered under that head.

72. Ex. 1. *A sphere whose centre of gravity is in its centre rolls down a perfectly rough inclined plane. Find the motion.*



Let α be the inclination of the plane to the horizon, a the radius of the sphere, mk^2 its moment of inertia about a horizontal diameter.

Let O be that point of the inclined plane which was initially touched by the sphere, and N the point of contact at the time t. Then it is obviously convenient to choose O for origin and ON for the axis of x.

The forces which act on the sphere are first the reaction R perpendicular to ON, secondly, F the friction acting at N along NO and mg acting vertically at C the centre.

The effective forces are $m\dfrac{d^2x}{dt^2}$, $m\dfrac{d^2y}{dt^2}$ acting at C parallel to the axes of x and y and a couple $mk^2\dfrac{d^2\theta}{dt^2}$ tending to turn the sphere round C in the direction NA. Here θ is the angle any fixed straight line in the body makes with a fixed straight line in space. We shall take the fixed straight line in the body to be the radius CA, and the fixed straight line in space the normal to the inclined plane. Then θ is the angle turned through by the sphere.

Resolving along and perpendicular to the inclined plane we have

$$m\frac{d^2x}{dt^2} = mg\sin\alpha - F \text{ .... } (1),$$

$$m\frac{d^2y}{dt^2} = -mg\cos\alpha + R \text{ .... } (2).$$

Taking moments about N to avoid the reactions, we have

$$ma\frac{d^2x}{dt^2} + mk^2\frac{d^2\theta}{dt^2} = mga\sin\alpha \text{ ...... } (3).$$

Since there are two unknown reactions F and R, we shall require two geometrical relations. Because there is no slipping at N, we have

$$x = a\theta \text{ .... } (4).$$

Also because there is no jumping

$$y = a \text{ .... } (5).$$

Both these equations are of the form described in the first method.

Differentiating (4) we get

$$\frac{d^2x}{dt^2} = a\frac{d^2\theta}{dt^2}.$$

Joining this to (3) we have

$$\frac{d^2x}{dt^2} = \frac{a^2}{a^2+k^2}g\sin\alpha \text{ .... } (6).$$

If the sphere be homogeneous, $k^2 = \frac{2}{5}a^2$, and we have

$$\frac{d^2x}{dt^2} = \frac{5}{7}g\sin\alpha.$$

If the sphere had been sliding down a *smooth* plane, the equation of motion would have been

$$\frac{d^2x}{dt^2} = g \sin \alpha,$$

so that two-sevenths of gravity is used in turning the sphere, and five-sevenths in urging the sphere downwards. Supposing the sphere to start from rest we have clearly

$$x = \frac{1}{2} \cdot \frac{5}{7} g \sin \alpha \,.\, t^2,$$

and the whole motion is determined.

In the above solutions, only a few of the equations of motion have been used, and if only the motion had been required it would have been unnecessary to write down any equations except (3) and (4). If the reactions also be required, we must use the remaining equations. From (1) we have

$$F = \frac{2}{7} mg \sin \alpha.$$

From (2) and (5) we have

$$R = mg \cos \alpha.$$

It is usual to delay the substitution of the value of k^2 in the equations until the end of the investigation, for this value is often very complicated. But there is another advantage. It serves as a verification of the signs in our original equations, for if equation (6) had been

$$\frac{d^2x}{dt^2} = \frac{a^2}{a^2 - k^2} g \sin \alpha,$$

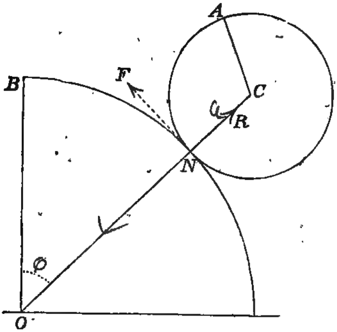
we should have expected some error to exist in the solution.

For it seems clear that the acceleration could not be made infinite by any alteration of the internal structure of the sphere.

73. Ex. 2. *A sphere rolls down another perfectly rough fixed sphere. Find the motion.*

Let a and b be the radii of the moving and fixed sphere, respectively, C and O the two centres.

Let OB be a vertical radius of the fixed sphere, and $\phi = \angle BOC$. Let F and R be the friction and the normal reaction at N. Then taking the tangential and normal resolutions, the equations to determine the motion of C are



$$(a+b)\frac{d^2\phi}{dt^2} = g \sin\phi - \frac{F}{m} \quad\text{.............. (1)},$$

$$(a+b)\left(\frac{d\phi}{dt}\right)^2 = g \cos\phi - \frac{R}{m} \quad\text{.............. (2)}.$$

Let A be that point of the moving sphere which originally coincided with B. Then if θ be the angle which any fixed line CA in the body makes with any fixed line in space as the vertical, we have by taking moments about C

$$\frac{d^2\theta}{dt^2} = \frac{Fa}{mk^2} \quad\text{...................... (3)}.$$

It should be observed that we cannot take θ as the angle ACO because, though CA is fixed in the body, CO is not fixed in space.

The geometrical equation is clearly

$$a(\theta - \phi) = b\phi \quad\text{.................. (4)}.$$

No other is wanted, since in forming equations (1) and (2) the constancy of the distance CO has been already supposed.

The form of equation (4) shews that we can apply the first method. We thus obtain

$$F = \frac{k^2}{k^2 + a^2} mg \sin\phi,$$

and we are finally led to the equation

$$(a+b)\frac{d^2\phi}{dt^2} = \frac{5}{7} g \sin\phi.$$

By multiplying by $2\dfrac{d\phi}{dt}$ and integrating we get after determining the constant

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{10}{7}\frac{g}{a+b}(1-\cos\phi),$$

the rolling body being supposed to start from rest at a point indefinitely near B.

To find where the body leaves the sphere we must put $R=0$. This gives by (2)

$$(a+b)\left(\frac{d\phi}{dt}\right)^2 = g\cos\phi;$$

$$\therefore\ \frac{10}{7} g(1-\cos\phi) = g\cos\phi;$$

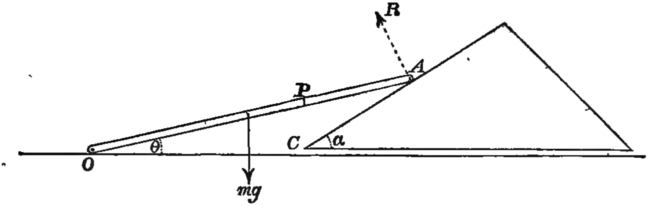
$$\therefore\ \cos\phi = \frac{10}{17}.$$

It may be remarked that this result is independent of the magnitudes of the spheres.

74. Ex. 3. *A rod OA can turn about a hinge at O, while the end A rests on a smooth wedge which can slide along a smooth horizontal plane through O. Find the motion.*

Let $\alpha =$ the inclination of the wedge, $M =$ its mass and $x = OC$.

Let $l =$ the length of the beam, $m =$ its mass and $\theta = AOC$. Let $R =$ the reaction at A. Then we have



the dynamical equations,

$$\frac{d^2x}{dt^2} = \frac{R \sin \alpha}{M} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{d^2\theta}{dt^2} = \frac{Rl \,.\, \cos(\alpha - \theta) - mg\frac{l}{2}\cos\theta}{mk^2} \quad\ldots\ldots\ldots (2),$$

and the geometrical equation,

$$x = \frac{l}{\sin \alpha} \,.\, \sin(\alpha - \theta) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It is obvious we must apply the second method of solution. Hence

$$2M\frac{dx}{dt}\,\frac{d^2x}{dt^2} + 2mk^2\frac{d\theta}{dt}\,\frac{d^2\theta}{dt^2} = -\,mgl\cos\theta\,\frac{d\theta}{dt}$$
$$+\,2R\left\{\sin\alpha\,\frac{dx}{dt} + l\cos(\alpha - \theta)\,\frac{d\theta}{dt}\right\}.$$

The coefficient of R is seen to vanish by differentiating equation (3). Integrating we have

$$M\left(\frac{dx}{dt}\right)^2 + mk^2\left(\frac{d\theta}{dt}\right)^2 = C - mgl\sin\theta.$$

Substituting from (3) we have

$$\left\{M\frac{l^2}{\sin^2\alpha}\cos^2(\alpha - \theta) + mk^2\right\}\left(\frac{d\theta}{dt}\right)^2 = C - mgl\sin\theta. \ldots (4).$$

If the beam start from rest when $\theta = \beta$, then $C = mgl\sin\beta$.

This equation cannot be integrated any further. We cannot therefore find θ in terms of t. But the angular velocity of the beam, and therefore the velocity of the wedge, is given by the above equation.

75. Ex. 4. *The bob of a heavy pendulum contains a spherical cavity which is filled with water. To determine the motion.*

Let O be the point of suspension, G the centre of gravity of the solid part of the pendulum, MK^2 its moment of inertia about O and let $OG=h$. Let C be the centre of the sphere of water, a its radius and $OC=c$. Let m be the mass of the water.

If we suppose the water to be a perfect fluid, the action between it and the case must, by the definition of a fluid, be normal to the spherical boundary. There will therefore be no force tending to turn the fluid round its centre of gravity. As the pendulum oscillates to and fro, the centre of the sphere will partake of its motion, but there will be no rotation of the water.

In order to avoid introducing into the equations the unknown reactions at the point of support, we shall take moments for the whole system about this point. Let θ be the angle CO, a fixed line in the body, makes with the vertical. Then, by Art. 51, the moment about O of the effective forces on the solid part of the pendulum is $MK^2\dfrac{d^2\theta}{dt^2}$. The effective forces on the water are equivalent to $-mc\left(\dfrac{d\theta}{dt}\right)^2$ and $mc\dfrac{d^2\theta}{dt^2}$ along and perpendicular to OC. Their moment about O is therefore $mc^2\dfrac{d^2\theta}{dt^2}$. The only impressed force is gravity, and its moment about O is $(Mh+mc)\,g\sin\theta$. The equation of motion is therefore

$$(MK^2+mc^2)\frac{d^2\theta}{dt^2}+(Mh+mc)\,g\sin\theta=0.$$

The length L' of the simple equivalent pendulum is, by Art. 54,

$$L'=\frac{MK^2+mc^2}{Mh+mc}.$$

Let mk^2 be the moment of inertia of the sphere of water about a diameter. Then if the water were to become solid and to be rigidly connected with the case, the length L of the simple equivalent pendulum would be, by similar reasoning,

$$L = \frac{MK^2 + m(c^2 + k^2)}{Mh + mc}.$$

It appears that $L' < L$, so that the time of oscillation is less than when the whole is solid.

On the strain at any point of a rod.

76. Suppose a rod OA to be in equilibrium under the action of any forces, it is required to determine the action across any section of the rod at P. This action may be conceived to be the resultant of the tensions positive or negative of the innumerable fibres which form the material of the rod. All these we know by Statics may be compounded into a single force R and a couple G acting at any point Q we may please to choose. Since each portion of the rod is in equilibrium, these must also be the resultants of all the external forces which act on the rod on *one side* of the section at P. If the section be indefinitely small it is usual to take Q in the plane of the section, and these two, the force R and the couple G, will together measure the "strain" at the section.

If the rod be bent by the action of the forces, the fibres on one side will all be stretched and on the other compressed. The rod will begin to break as soon as these fibres have been sufficiently stretched or compressed. Let us compare the tendencies of the force R and the couple G to break the rod. Let A be the area of the section of the rod, then a force F pulling the rod will cause a resultant force $R = F$, and will produce a tension in the fibres which when referred to a unit of area is equal to $\frac{F}{A}$. The same force F acting on the rod at an arm from P whose length is p, will cause a couple $G = Fp$, which must be balanced by the couple formed by the tensions. Let $2a$ be the mean breadth of the rod, then the mean tension referred to a unit of area produced by G is of the order $\frac{F}{A} \cdot \frac{p}{a}$. Now if the section of the rod be very small $\frac{p}{a}$ will be large. It appears therefore that the couple, when it exists, will generally have much more effect in breaking the rod than the force. This couple is therefore often taken to measure the whole effect of the forces to break the rod. The "*tendency to break*" at any point P of a rod OA of very small section is measured by the moment about P of all the forces which act on either of the sides OP or PA of the rod.

If the rod be in motion the same reasoning will, by D'Alembert's principle, be applicable; provided we include the reversed effective forces among the forces which act on the rod.

In most cases the rod will be so little bent that in finding the moment of the impressed forces we may neglect the effects of curvature.

If the section of the rod be not very small, this measure of the "Tendency to break" becomes inapplicable. It then becomes necessary to consider both the force and the couple. This does not come within the limits of the present treatise, and the reader is referred to works on Elastic Solids.

In the case of a string the couple vanishes and the force acts along a tangent to the string. The strain at any point is therefore simply measured by the tension.

77. To find the "Tendency to break" at any point P of the rod OA in the problem of Art. 74.

Let du be any element of the rod distant u from P and on the side of P nearer the end A of the rod, and let $PA = z$. The effective forces on du are

$$m\frac{du}{l}(u+l-z)\frac{d^2\theta}{dt^2} \text{ and } -m\frac{du}{l}(u+l-z)\left(\frac{d\theta}{dt}\right)^2,$$

respectively perpendicular to, and along the rod. The impressed force is $m\frac{du}{l}g$. The effective forces being reversed, the tendency to break at P is equal to the moment about P of all the forces which act on the part PA of the rod. If this be called L, we have

$$L=\int_0^z\left\{m\frac{du}{l}(u+l-z)\,u\frac{d^2\theta}{dt^2}+m\frac{du}{l}gu\cos\theta\right\}-Rz\cos(\alpha-\theta)$$

$$=\frac{m}{l}\cdot\left\{\left(\frac{lz^2}{2}-\frac{z^3}{6}\right)\frac{d^2\theta}{dt^2}+g\frac{z^2}{2}\cos\theta\right\}-Rz\cos(\alpha-\theta)\dots\dots\dots(5),$$

and the values of $\frac{d^2\theta}{dt^2}$ and R must be substituted.

To find where the rod, supposed equally strong throughout, is most likely to break, we must make L a maximum. This gives $\frac{dL}{dz}=0$;

$$\therefore\ \frac{m}{l}\left\{\left(lz-\frac{z^2}{2}\right)\frac{d^2\theta}{dt^2}+gz\cos\theta\right\}=R\cos(\alpha-\theta).$$

Substituting for R from equation (2) we get

$$\left(lz-\frac{z^2}{2}-k^2\right)\frac{d^2\theta}{dt^2}=g\cos\theta\left(\frac{l}{2}-z\right)\dots\dots\dots(6).$$

But from equation (4) we have

$$\left(\frac{d\theta}{dt}\right)^2=\frac{mgl(\sin\beta-\sin\theta)}{M\frac{l^2}{\sin^2\alpha}\cos^2(\alpha-\theta)+mk^2}.$$

Therefore differentiating with respect to t and dividing by $\frac{d\theta}{dt}$ we have

$$2\frac{d^2\theta}{dt^2}=-mgl.\frac{mk^2\cos\theta+M\frac{l^2}{\sin^2\alpha}.\cos(\alpha-\theta).P}{\left\{M\frac{l^2}{\sin^2\alpha}\cos^2(\alpha-\theta)+mk^2\right\}^2},$$

where $P=\cos\alpha+\sin(\alpha-\theta)\,(2\sin\beta-\sin\theta)$.

Substituting this in equation (6) we have a quadratic to find z.

On friction between imperfectly rough bodies.

78. When one part of a body rests on another a force is called into play tending to prevent slipping. This force is called friction. Its direction is always opposite to that of the relative motion of the points in contact. Its magnitude is just sufficient to prevent motion. But no more than a certain amount can be called into play, if more be required to prevent relative motion, motion will ensue. This amount is called "limiting friction." The amount of limiting friction is found by experiment to bear a constant ratio to the magnitude of the normal pressure. This ratio is called the coefficient of friction. It is independent of the velocity of the body and the extent of the surfaces in contact. If however the bodies have remained in contact for some time in a position of equilibrium, it is found that the coefficient of friction is a little greater than it is after motion has begun. The magnitude of the coefficient of friction depends only on the nature of the bodies in contact, and may be taken as a proper measure of their mutual roughness.

There is a certain discontinuity about friction. When two bodies touch each other, it will generally be part of the problem to determine whether the friction is limiting friction or not. Even when we have discovered the nature of the initial friction, we shall have to watch for sudden changes. If when a body is rolling the magnitude of the friction should approach its limit, or if when a body is sliding the velocities of the points of contact should approach equality, we may expect a change in the nature of the friction. We may generally determine whether the friction is limiting or not in the following manner.

Assume F to be the friction required to make the bodies roll on each other. Let R be the normal reaction between them, and μ the coefficient of friction. Then by writing down the equations of motion and solving them by the methods indicated in Art. 71 we may determine both F and R. If

$\frac{F}{R}$ be not greater than μ, enough friction can be called into play to make the bodies roll on each other. In this case therefore we have obtained the true equations of motion. But if $\frac{F}{R}$ be greater than μ the bodies are not sufficiently rough to secure rolling. The relative motion will therefore be a sliding motion, possibly accompanied by rotations of each body. In this case the equations do not represent the true motion. To correct them we must replace F by μR and remove the geometrical equation, which expressed the fact that there is no slipping between the bodies. The whole form of the solution will probably be changed.

If the bodies be perfectly rough μ is infinite, and there is no limit to the amount of friction which can be called into play. There can therefore be no sliding between the bodies.

We have spoken only of the friction which tends to prevent one body from sliding over another. But when one body rolls on another without any tendency to rub, there is another cause of resistance, which arises rather from the cohesion of the surfaces than from friction properly so called. This is much less than the former, and is usually neglected.

79. Ex. 1. *A sphere is placed at rest on a rough inclined plane, the coefficient of friction being μ, determine whether the sphere will slide or roll.*

Let F be the friction required to make the sphere roll. The problem then becomes the same as that discussed in Art. 72. We have therefore

$$\frac{F}{R} = \frac{2}{7} \tan \alpha,$$

where α is the inclination of the plane to the horizon.

If then $\frac{2}{7} \tan \alpha$ be not greater than μ, the solution given in the article referred to is the correct one. But if $\mu < \frac{2}{7} \tan \alpha$

the sphere will begin to slide on the inclined plane. The subsequent motion will be given by the equations

$$m\frac{d^2x}{dt^2} = mg\sin\alpha - \mu R \text{............}(1),$$

$$0 = -mg\cos\alpha + R \text{............}(2),$$

$$ma\frac{d^2x}{dt^2} + mk^2\frac{d^2\theta}{dt^2} = mga\sin\alpha \text{............}(3),$$

whence we have, remembering that $k^2 = \frac{2}{5}a^2$,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= g(\sin\alpha - \mu\cos\alpha) \\ \frac{d^2\theta}{dt^2} &= \frac{5}{2}\mu\frac{g}{a}\cos\alpha \end{aligned}\right\}.$$

Since the sphere starts from rest, we have by integration

$$\left.\begin{aligned} x &= \frac{1}{2}gt^2(\sin\alpha - \mu\cos\alpha) \\ \theta &= \frac{5}{4}\mu\frac{g}{a}t^2\cos\alpha \end{aligned}\right\}.$$

The velocity of the point of the sphere in contact with the plane is

$$\frac{dx}{dt} - a\frac{d\theta}{dt} = gt\left(\sin\alpha - \frac{7}{2}\mu\cos\alpha\right).$$

But since, by hypothesis, μ is less than $\frac{2}{7}\tan\alpha$, this velocity can never vanish. The friction therefore will never change to rolling friction. The motion has thus been completely determined.

80. Ex. 2. *A sphere is rotating about a horizontal diameter, and is gently placed on a rough horizontal plane, the coefficient of friction being μ. Determine the subsequent motion.*

Since the velocity of the point of contact with the horizontal plane is not zero, the sphere will evidently begin to slide, and the motion of its centre will be along a straight line perpendicular to the initial axis of rotation. Let this straight line be taken as the axis of x, and let θ be the angle between the vertical and that radius of the sphere which was initially vertical. Let a be the radius of the sphere, mk^2 its moment of inertia about a diameter, and Ω the initial angular velocity. Let R be the normal reaction of the plane. Then the equations of motion are clearly

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \mu R \\ 0 &= mg - R \\ mk^2\frac{d^2\theta}{dt^2} &= -\mu Ra \end{aligned}\right\}\text{............(1)},$$

whence we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mu g \\ \frac{d^2\theta}{dt^2} &= -\frac{5}{2}\mu\frac{g}{a} \end{aligned}\right\}\text{...............(2)}.$$

Integrating, and remembering that the initial value of $\frac{d\theta}{dt}$ is Ω, we have

$$\left.\begin{aligned} x &= \frac{1}{2}\mu g t^2 \\ \theta &= \Omega t - \frac{5}{4}\mu\frac{g}{a}t^2 \end{aligned}\right\}\text{.............(3)}.$$

But it is evident that these equations cannot represent the whole motion, for they would make $\frac{dx}{dt}$, the velocity of the centre of the sphere, increase continually. This is quite contrary to experience.

The velocity of the point of the sphere in contact with the plane is

$$\frac{dx}{dt} - a\frac{d\theta}{dt} = -a\Omega + \frac{7}{2}\mu g t.$$

This vanishes at a time

$$t_1 = \frac{2}{7}\frac{a\Omega}{\mu g} \dots\dots\dots\dots (4).$$

At this instant the friction suddenly changes its character. It now becomes only of sufficient magnitude to keep the point of contact of the sphere at rest.

Let F be the friction required to effect this. The equations of motion will then be

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= F \\ 0 &= mg - R \\ mk^2\frac{d^2\theta}{dt^2} &= -Fa \end{aligned}\right\} \dots\dots\dots\dots (5),$$

and the geometrical equation will be

$$x = a\theta.$$

Differentiating this twice, and substituting from the dynamical equations, we get

$$F(a^2 + k^2) = 0.$$

Therefore $$F = 0.$$

That is, no friction is required to keep the point of contact of the sphere at rest, and therefore none will be called into play. The sphere will therefore move uniformly with the velocity which it had at the time t_1. Substituting the value of t_1 in the expression for $\frac{dx}{dt}$ obtained from equations (3) we find that this velocity is $\frac{2}{7}a\Omega$. It appears therefore that the sphere will move with a uniformly increasing velocity for a time $\frac{2}{7}\frac{a\Omega}{\mu g}$ and will then move uniformly with a velocity $\frac{2}{7}a\Omega$. It may be remarked that this velocity is independent of μ.

If the plane be perfectly rough, μ is infinite, and the time t_1 vanishes. The sphere therefore immediately begins to move with a uniform velocity $= \frac{2}{7}a\Omega$.

The effect of the cohesion of the sphere to the plane is to diminish the angular velocity. The velocity of the lowest point will therefore tend to be no longer equal to zero, a sliding friction will then be called into play. The combined effect of these two will gradually reduce the sphere to rest.

On Impulsive Forces.

81. In the case in which the impressed forces are impulsive the general principle enûnciated in Art. 69 of this chapter requires but slight modification.

Let (u, v), (u', v') be the velocities of the centre of gravity of any body of the system resolved parallel to any rectangular axes respectively just before and just after the action of the impulses. Let ω and ω' be the angular velocities of the body about the centre of gravity at the same instants. And let Mk^2 be the moment of inertia of the body about the centre of gravity. Then the effective forces on the body are equivalent to two forces measured by $M(u'-u)$ and $M(v'-v)$ acting at the centre of gravity parallel to the axes of coordinates together with a couple measured by $Mk^2(\omega'-\omega)$.

The resultant effective forces of all the bodies of the system may be found by the same rule. By D'Alembert's principle these will be in equilibrium with the impressed forces. The equations of motion may then be found by resolving in such directions and taking moments about such points as may be found most convenient.

In many cases it will be found that by the use of Virtual Velocities the elimination of the unknown reactions may be effected without difficulty.

82. *Two bodies impinge on each other, to explain the nature of the action which takes place between them.*

When two spheres of any hard material impinge on each other, they appear to separate almost immediately, and a finite change of velocity is generated in each by their mutual action. This sudden change of velocity is the characteristic of an impulsive force. Let the centres of gravity of the spheres be moving before impact in the same straight line with velocities u and v. Then after impact they will continue to move in the same straight line, and let u', v' be the velocities. Let m, m' be the masses of the spheres, R the action between them, then we have by the last Article,

$$\left.\begin{aligned} u' - u &= -\frac{R}{m} \\ v' - v &= \frac{R}{m'} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

These equations are not sufficient to determine the three quantities u', v', R. To obtain a third equation we must consider what takes place during the impact.

Each of the balls will be slightly compressed by the other, so that they will no longer be perfect spheres. Each also will, in general, tend to return to its original shape so that there will be a rebound. The period of impact may therefore be divided into two parts. First, the period of compression, while the distance between the centres of gravity of the two bodies is diminishing, and secondly the period of restitution while the distance between the centres of gravity is increasing. At the termination of this second period the bodies separate.

The arrangement of the particles of a body being disturbed by impact, we ought to determine the relative motions of the several parts of the body. Thus we might regard each body as a collection of free particles connected by their mutual actions. These particles being thus set in motion might continue always in motion oscillating about some mean positions.

It is however usual to assume that the changes of shape and structure are so small that the effect in altering the position of the centre of gravity and the moments of inertia of the body may be neglected, and that the whole time of impact is so short that the motion of the body in that time may be neglected. If for any bodies these assumptions are not true, the effects of their impact must be deduced from the equations of the second order. We may therefore assert that at the moment of greatest compression the centres of gravity of the two bodies are moving with equal velocity.

The ratio of the magnitude of the action between the bodies during the period of restitution to that during compression is found to be different for bodies of different materials. In some cases this ratio is so small that the force during the period of restitution may be neglected. The bodies are then said to be *inelastic.* In this case we have just after the impact

$$u' = v'.$$

This gives

$$R = \frac{mm'}{m + m'} (u - v),$$

whence

$$u' = \frac{mu + m'v}{m + m'}.$$

If the force of restitution cannot be neglected, let R be the whole action between the balls, R_0 the action up to the moment of greatest compression. The magnitude of R must be found by experiment. This may be done by determining the values of u' and v', and thus determining R by means of equations (1). These experiments were made in the first instance by Newton, and the result is that $\frac{R}{R_0}$ is a *constant ratio* depending on the material of the balls. Let this constant ratio be called $1 + e$. The quantity e is always less than unity, in the limiting case when $e = 1$ the bodies are said to be perfectly elastic.

The value of e being supposed known the velocities after impact may be easily found. The action R_0 must be first calculated as if the bodies were inelastic, then the whole value of R may be found by multiplying this result by $1 + e$. This gives

$$R = \frac{mm'}{m + m'} (u - v) (1 + e),$$

whence u' and v' may be found by equations (1).

83. Ex. 1. *A string is wound round the circumference of a circular reel, and the free end is attached to a fixed point. The reel is then lifted up and let fall so that at the moment when the string becomes tight it is vertical, and a tangent to the reel. The whole motion being supposed to take place in one plane, determine the effect of the impulse.*

The reel in the first instance falls vertically without rotation. Let v be the velocity of the centre at the moment when the string becomes tight; v', ω' the velocity of the centre and the angular velocity just after the impulse. Let T be the impulsive tension, mk^2 the moment of inertia of the reel about its centre of gravity, a its radius.

In order to avoid introducing the unknown tension into the equations of motion, let us take moments about the point of contact of the string with the reel; we then have

$$m(v'-v)\,a + mk^2\omega' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots \;(1).$$

At the moment of greatest compression the part of the reel in contact with the string has no velocity.

Hence $$v' - a\omega' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(2).$$

Solving these we have $\omega' = \dfrac{av}{a^2+k^2}$.

If the reel be a homogeneous cylinder $k^2 = \dfrac{a^2}{2}$, and in this case we have $$\omega' = \frac{2}{3}\frac{v}{a}, \quad v' = \frac{2}{3}v.$$

If it be required to find the impulsive tension, we have resolving vertically

$$m(v'-v) = -T \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(3).$$

Hence $$T = \frac{1}{3}mv.$$

These results will give the initial motion after the string has become tight if the string and reel be inelastic. If they have a common elasticity e, then we know that the correct value of T is

$$T = \frac{1}{3}mv(1+e).$$

The motion of a reel acted on by this *known* impulsive force is easily found.

Resolving vertically we have

$$m(v' - v) = -\frac{1}{3} mv (1 + e).$$

Taking moments about the centre of gravity

$$mk^2\omega' = \frac{1}{3} mva (1 + e),$$

whence v' and ω' may be found.

To find the subsequent motion. The centre of the reel *begins* to descend vertically, and there is no horizontal force on it. Hence it will continue to descend in a vertical straight line, and throughout all the subsequent motion the string is vertical. The motion may therefore be easily investigated as in Art. 72. If we put $\alpha = \frac{\pi}{2}$, and let $F =$ the finite tension of the string, it may be shown that $F =$ one-third of the weight, and that the reel descends with a uniform acceleration $= \frac{2}{3} g$. The initial velocity of the reel has been found in this article $= v'$, so that the space descended in a time t after the impact is $= v't + \frac{1}{2} \cdot \frac{2}{3} gt^2$.

84. Ex. 2. *An inelastic spherical ball, moving without rotation on a smooth horizontal plane, impinges with velocity v against a rough vertical wall whose coefficient of friction is μ. The line of motion of the centre of gravity before incidence making an angle α with the normal to the wall, determine the motion after impact.*

Let v_x, v_x' be the velocities of the centre of the ball just before and just after impact resolved along the wall, v_y, v_y' the velocities resolved perpendicular to the wall in such directions that v_x and v_y are positive. Then $v_x = v \sin \alpha$, $v_y = v \cos \alpha$; let ω' be the angular velocity of the ball, let m be its mass, and a its radius.

Let R be the normal blow, then the impulsive friction cannot be greater than μR. Let the friction be F.

The equations of motion are

$$v_x' - v_x = -\frac{F}{m} \quad \text{......................} (1),$$

$$v_y' - v_y = -\frac{R}{m} \quad \text{......................} (2),$$

$$\omega' = \frac{Fa}{mk^2} \quad \text{......................} (3).$$

At the moment of greatest compression the velocity of the point of the ball in contact with the wall must be zero. Hence we have

$$\left.\begin{aligned} v_x' - a\omega' &= 0 \\ v_y' &= 0 \end{aligned}\right\} \quad \text{......................} (2).$$

This gives

$$\left.\begin{aligned} R &= mv \cos \alpha \\ F &= m \frac{k^2}{a^2 + k^2} v \sin \alpha \end{aligned}\right\} \quad \text{..................} (3),$$

whence v_x' and ω' can be found.

This result is only true provided F is not greater than μR, that is $\frac{k^2}{a^2 + k^2} \tan \alpha$ not greater than μ.

If α be so great that this inequality does not hold, we must have $F = \mu R$. The equations (2) now reduce to the single one

$$v_y' = 0 \text{..........................} (4),$$

because it is no longer true that the sliding motion of the ball along the wall is destroyed. This equation gives the same value of R as before, and

$$\therefore\ F = \mu mv \cos \alpha \text{....................} (5),$$

whence v_x' and ω' can be found.

85. Ex. 3. *Four equal rods each of length* 2a *and mass* m *are freely jointed so as to form a rhombus. The system falls from rest with a diagonal vertical under the action of gravity and strikes against a fixed horizontal inelastic plane. Find the subsequent motion.*

Let AB, BC, CD, DE be the rods and let AC be the vertical diagonal impinging on the horizontal plane at A. Let V be the velocity of every point of the rhombus just before impact and let α be the angle any rod makes with the vertical.

Let u, v be the horizontal and vertical velocities of the centre of gravity and ω the angular velocity of either of the upper rods just after impact. Then the effective forces on either rod are equivalent to the force $m(v-V)$ acting vertically and mu horizontally at the centre of gravity and a couple $mk^2\omega$ tending to increase the angle α. Let R be the impulse at C, the direction of which by the rule of symmetry is horizontal. To avoid introducing the reaction at B into our equations, let us take moments for the rod BC about B and we have

$$mk^2\omega + m(v-V)a\sin\alpha - mua\cos\alpha = -R\,.\,2a\cos\alpha \;\ldots\ldots (1).$$

Each of the lower rods will begin to turn round its extremity A as a fixed point. If ω' be its angular velocity just after impact, the moment of the momentum about A just after impact will be $m(k^2+a^2)\,\omega'$ and just before will be $mVa\sin\alpha$. The difference of these two is the moment about A of the effective forces on either of the lower rods. We may now take moments about A for the two rods AB, BC together and we have

$$m(k^2+a^2)\,\omega' - mVa\sin\alpha - mk^2\omega + m(v-V)a\sin\alpha + mu\,.\,3a\cos\alpha$$
$$= R\,.\,4a\cos\alpha \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The geometrical equations may be found thus.

Since the two rods must make equal angles with the vertical during the whole motion we have

$$\omega' = \omega \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Again, since the two rods are connected at B the velocities of the extremities of the two rods must be the same in direction and magnitude. Resolving these horizontally and vertically, we have

$$u + a\omega\cos\alpha = 2a\omega'\cos\alpha \dots\dots\dots\dots\dots\dots (4),$$

$$v - a\omega\sin\alpha = 2a\omega'\sin\alpha \dots\dots\dots\dots\dots\dots (5).$$

These five equations are sufficient to determine the initial motion.

Eliminating R between (1) and (2), substituting for u, v, ω' in terms of ω from the geometrical equations, we find

$$\omega = \frac{3}{2} \cdot \frac{V\sin\alpha}{a(1+3\sin^2\alpha)} \dots\dots\dots\dots\dots\dots (6).$$

In this problem we might have avoided the introduction of the unknown reaction R by the use of Virtual Velocities. Suppose we give the system such a displacement that the inclination of each rod to the vertical is increased by the same quantity $\delta\alpha$. Then the principle of Virtual Velocities gives

$$\left.\begin{array}{l} mk^2\omega\delta\alpha - m(v-V)\,\delta(3a\cos a) + mu\delta(a\sin\alpha) \\ \quad + m(k^2+a^2)\,\omega'\delta\alpha + mV\delta(a\cos\alpha) \end{array}\right\} = 0,$$

which reduces to

$$(2k^2+a^2)\,\omega - Va\sin\alpha + 3(v-V)\,a\sin\alpha + ua\cos\alpha = 0,$$

and the solution may be continued as before.

To find the subsequent motion. This may be found very easily by the method which will be given in the chapter on *Vis Viva.* But we may also proceed thus. The effective forces on either of the upper rods will be represented by the differential coefficients $m\frac{dv}{dt}$, $m\frac{du}{dt}$, $mk^2\frac{d\omega}{dt}$, and the moment for either of the lower rods will be $m(k^2+a^2)\frac{d\omega'}{dt}$. Let θ be the angle any rod makes with the vertical at the time t. Taking moments in the same way as before, we have

$$mk^2\frac{d\omega}{dt} + m\frac{dv}{dt}a\sin\theta - m\frac{du}{dt}a\cos\theta$$

$$= -R \,.\, 2a\cos\theta + mga\sin\theta \dots\dots\dots\dots\dots\dots (1)',$$

$$m(k^2+a^2)\frac{d\omega'}{dt}-mk^2\frac{d\omega}{dt}+m\frac{dv}{dt}a\sin\theta+m\frac{du}{dt}\cdot 3a\cos\theta$$
$$=R\cdot 4a\cos\theta+2mga\sin\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots (2)'.$$

The geometrical equations are the same as those given above, with θ written for α.

Eliminating R and substituting for u v, we get

$$(2k^2+a^2)\frac{d\omega}{dt}+a^2\left\{9\sin\theta\frac{d}{dt}(\omega\sin\theta)+\cos\theta\frac{d}{dt}(\omega\cos\theta)\right\}$$
$$=4ga\sin\theta;$$

multiplying both sides by $\omega=\dfrac{d\theta}{dt}$ and integrating, we get

$$\{2(k^2+a^2)+8a^2\sin^2\theta\}\,\omega^2=C-8ga\cos\theta.$$

Initially when $\theta=\alpha$, ω has the value given by equation (6). Hence we find that the angular velocity ω when the inclination of any rod to the vertical is θ is given by

$$(1+3\sin^2\theta)\,\omega^2=\frac{9V^2}{4a^2}\cdot\frac{\sin^2\alpha}{1+3\sin^2\alpha}+\frac{3g}{a}(\cos\alpha-\cos\theta).$$

86. Ex. 4. *A free lamina of any form is turning in its own plane about an instantaneous centre of rotation* S *and impinges on an obstacle at* P, *situated in the straight line joining the centre of gravity* G *to* S. *To find the point* P *when the magnitude of the blow is a maximum.*

First, let the obstacle P *be a fixed point.*

Let $GP=x$, and let R be the force of the blow. Let $SG=h$, and let ω, ω' be the angular velocities about the centre of gravity before and after the impact. Then $h\omega$ is the linear velocity of G just before the impact; let v' be its linear velocity just after the impact.

We have the equations

$$\left.\begin{aligned}\omega'-\omega&=\frac{-Rx}{Mk^2}\\ v'-h\omega&=-\frac{R}{M}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and supposing the point of impact to be reduced to rest,

$$v' + x\omega' = 0 \quad\text{.........................}\quad (2).$$

Substituting for ω' and v' from (1) in equation (2), we get

$$R = M\omega \,.\, k^2 \frac{x+h}{x^2+k^2}.$$

This is to be made a maximum. Equating to zero its differential coefficient with respect to x, we get

$$x^2 + 2hx - k^2 = 0 \quad\text{....................}\quad (3);$$

$$\therefore\ x = -h \pm \sqrt{h^2+k^2}.$$

One of these values of x is positive and the other negative. Both these correspond to *maximum* points of percussion, but opposite in direction. Thus there is a point P with which the body strikes in front not only more forcibly than with the centre of percussion O itself, but also more forcibly than with any other point; and at the same time there is another point P' with which the body strikes with the greatest possible force, but it does so in the rear of its own translation through space*.

Let k' be the radius of gyration about the instantaneous axis of rotation, then $h^2+k^2=k'^2$ and $h+x=SP$; $\therefore\ SP=\pm k'$. Hence the two points P, P' are at equal distances from S. Also if O be the centre of oscillation with respect to S as a centre of suspension, $SG \,.\, SO = k'^2$; $\therefore\ SP^2 = SG \,.\, SO$.

Since GP, GP' are the roots of the quadratic equation (3),

$$\therefore\ \left.\begin{aligned} GP' - GP &= 2h \\ GP \,.\, GP' &= k^2 \end{aligned}\right\}.$$

The latter equation shows that if P be made a point of suspension, P' is the corresponding centre of oscillation. It is easy to see that PP' is harmonically divided in G and O.

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* Poinsot, Sur la percussion des corps, Liouville's *Journal*, 1857; translated in the *Annals of Philosophy*, 1858.

Secondly, let the obstacle be a free particle of mass m.

Then, besides the equations (1), we have the equation of motion of the particle m. Let V' be its velocity after impact,

$$\therefore\ V' = \frac{R}{m} \text{ ....} (4).$$

The point of impact in the two bodies will have after impact the same velocity, hence instead of equation (2) we have

$$V' = v' + x\omega' \text{ ....} (5).$$

Substituting for ω', v', V' from equations (1) and (4) in equation (5), we get

$$R = M\omega \,.\, k^2 \,.\, \frac{m\,(x+h)}{(M+m)\,k^2 + mx^2}.$$

This is to be made a maximum. Equating to zero its differential coefficient with respect to x, we get

$$x^2 + 2hx = k^2\left(1 + \frac{M}{m}\right) \text{ ....} (6);$$

$$\therefore\ x = -h \pm \sqrt{h^2 + k^2\left(1 + \frac{M}{m}\right)}.$$

This point does not coincide with that found when the obstacle was fixed, unless m is infinite. To find when it coincides with the centre of oscillation, we must put $k^2 = xh$. This gives $\frac{M}{m} = \frac{x+h}{h}$, or if $l = x + h$ be the length of the simple equivalent pendulum, $\frac{M}{m} = \frac{l}{h}$.

Since $V' = \frac{R}{m}$, it is evident that when R is a maximum V' is a maximum. Hence the two points found by equation (6) might be called the centres of greatest communicated velocity.

There are other singular points in a moving body whose positions may be found; thus we might inquire at what point a body must impinge against a *fixed* obstacle, that *first* the linear velocity of the centre of gravity might be a maximum, or *secondly*, that the angular velocity might be a maximum. These points, respectively, have been called by Poinsot the centres of maximum Reflexion and Conversion. Referring to equations (1), we see that when v' is a maximum R is either a maximum or a minimum, and hence it may be shown that the first point coincides with the point of greatest impact. When ω' is a maximum, we have to make

$$\omega - \frac{Rx}{Mk^2} = maximum.$$

Substituting for R, this gives

$$x^2 - 2\frac{k^2}{h}x - k^2 = 0 \qquad (7).$$

If O be the centre of oscillation, we have $GO = \frac{k^2}{h}$. Let this length be represented by h'. Then the equation (7) becomes

$$x^2 - 2h'x - k^2 = 0 \qquad (8).$$

The roots of this equation are the same functions of h' and k that those of equation (3) are of h and k, except that the signs are opposite. Now S and O are on opposite sides of G, hence the positions of the two centres of maximum Conversion bear to O and G the same relation that the positions of the two centres of maximum Reflexion do to S and G. If the point of suspension be changed from S to O, the positions of the centres of maximum Reflexion and Conversion are interchanged.

Initial Motions.

87. Suppose a system of bodies to be in equilibrium and that one of the supports suddenly gives way. It is required to find the initial motion of the bodies and the initial values of the reactions which exist between the several bodies.

The problem of finding the initial motion of a dynamical system is the same as that of expanding the co-ordinates of the moving particles in powers of the time t. Let (x, y, θ) be the co-ordinates of any body of the system. For the sake of brevity let us denote by accents differential coefficients with regard to the time, and let the suffix zero denote initial values. Thus x_0'' denotes the initial value of $\frac{d^2x}{dt^2}$. By Taylor's theorem we have

$$x = a + x_0'' \frac{t^2}{\underline{|2}} + x_0''' \frac{t^3}{\underline{|3}} + \dots\dots\dots\dots\dots\dots \quad (1),$$

the term x_0' is omitted because we shall suppose the system to start from rest.

First, let only the initial values of the reactions be required. The dynamical equations will contain the coordinates, their second differential coefficients with regard to t, and the unknown reactions. There will be as many geometrical equations as reactions. From these we have to eliminate the second differential coefficients and find the reactions. The process will be as follows.

Write down the geometrical equations, differentiate each twice and then simplify the results by substituting for the coordinates their initial values. Thus, if we use Cartesian coordinates, let $\phi(x, y, \theta) = 0$ be any geometrical relation, we have since $x_0' = 0$, $y_0' = 0$, $\theta_0' = 0$,

$$\frac{d\phi}{dx} x_0'' + \frac{d\phi}{dy} y_0'' + \frac{d\phi}{d\theta} \theta_0'' = 0.$$

The process of differentiating the equations may sometimes be much simplified when the origin has been so chosen that the initial values of some at least of the coordinates are zero. We may then simplify the equations by neglecting the squares and products of all such coordinates. For if we have a term x^2, its second differential coefficient is $\frac{1}{2}(xx'' + x'^2)$, and if the initial value of x is zero, this vanishes.

The geometrical equations must be obtained by supposing the bodies to have their *displaced* position, because we require to differentiate them. But this is not the case with the dynamical equations. These we may write down on the supposition that each body is in its *initial* position. These equations may be obtained according to the rules given in page 96. The forms there given for the effective forces admit in this problem of some simplifications. Thus since $r_0' = 0$, $\phi_0' = 0$, the acceleration along and perpendicular to the radius vector take the simple forms r_0'' and $r\phi_0''$. So again the acceleration $\frac{v^2}{\rho}$ along the normal vanishes. If, for example, we know the initial direction of motion of the centre of gravity of any one of the bodies, we might conveniently resolve along the normal to the path. This will supply an equation which contains only the impressed forces and such tensions or reactions as may act on that body. If there be only one reaction, this equation will suffice to determine its initial value.

We may also deduce from the equations the values of x_0'', y_0'', θ_0'', and thus by substituting in equation (1) we have found the initial motion up to terms depending on t^2.

88. *Secondly, let the initial motion be required.* How many terms of the series (1) it may be necessary to retain will depend on the nature of the problem. Suppose the radius of curvature of the path described by the centre of gravity of one of the bodies to be required. We have

$$\rho = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y'' - y'x''},$$

and by differentiating equation (1)

$$x' = x_0'' t + x_0''' \frac{t^2}{\lfloor 2} + x_0^{\text{iv}} \frac{t^3}{\lfloor 3} + \dots$$

$$x'' = x_0'' + x_0''' t + x_0^{\text{iv}} \frac{t^2}{\lfloor 2} + \dots$$

$$\&\text{c.} = \&\text{c.};$$

$$\therefore\ (x'^2 + y'^2)^{\frac{3}{2}} = (x_0''^2 + y_0''^2)^{\frac{3}{2}} t^3 + \dots$$

$$x'y'' - y'x'' = (x_0'' y_0''' - x_0''' y_0'') \frac{t^2}{2} + (x_0'' y_0^{\text{iv}} - x_0^{\text{iv}} y_0'') \frac{t^3}{3} + \dots$$

results which may also be obtained by a direct use of Taylor's theorem.

If then the body start from rest, the radius of curvature is zero. But if $x_0''y_0''' - x_0'''y_0'' = 0$, we have

$$\rho = 3\,\frac{(x_0''^2 + y_0''^2)^{\frac{3}{2}}}{x_0''y_0^{\text{iv}} - x_0^{\text{iv}}y_0''}.$$

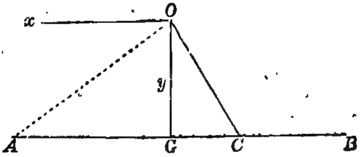
To find these differential coefficients we may proceed thus. Differentiate each dynamical equation twice and then reduce it to its initial form by writing for x, y, θ, &c. their initial values, and for x', y', θ' zero. Differentiate each geometrical equation four times and then reduce each to its initial form. We shall thus have sufficient equations to determine x_0'', x_0''', x_0^{iv}, &c., R_0, R_0', R_0'', &c., where R is any one of the unknown reactions. It will often be an advantage to eliminate the unknown reactions from the equations *before* differentiation. We shall then have only the unknown coefficients x_0'', x_0''', &c. entering into the equations.

If we know the direction of motion of one of the centres of gravity under consideration, we can take the axis of y a tangent to its path. Then we have $\rho = \frac{y^2}{x}$, where x is of the second order, y of the first order, of small quantities. We may therefore neglect the squares of x and the cubes of y. This will greatly simplify the equations. If the body start from rest we have $x_0' = 0$, and if $x_0''' = 0$, we may then use the formula

$$\rho = 3\,\frac{y_0''^2}{x_0^{\text{iv}}}.$$

89. Ex. *A circular disc is hung up by three equal strings attached to three points at equal distances in its circumference, and fastened to a peg vertically over the centre of the disc. One of these strings is suddenly cut. Determine the initial circumstances of motion.*

Let O be the peg, AB the circle seen by an eye in its



plane. Let OA be the string which is cut and C be the middle point of the chord joining the points of the circle to which the two other strings are attached. Then the two tensions, each equal to T, are throughout the motion equivalent to a resultant tension R along CO. If 2α be the angle between the two strings, we have

$$R = 2T\cos\alpha.$$

Let l be the length of OC, β be the angle GOC, a be the radius of the disc. Let (x, y) be the co-ordinates of the *displaced* position of the centre of gravity with reference to the origin O, x being measured horizontally to the left and y vertically downwards. Let θ be the angle the displaced position of the disc makes with AB.

By drawing the disc in its displaced position it will be seen that the co-ordinates of the displaced position of C are $x - l\sin\beta\cos\theta$, and $y - l\sin\beta\sin\theta$. Hence since the length OC remains constant and equal to l we have

$$x^2 + y^2 - 2l\sin\beta\,(x\cos\theta + y\sin\theta) = l^2\cos^2\beta.$$

Suppose the initial tensions only to be required. It will be sufficient to differentiate this twice. Since we may neglect the squares of small quantities, we may omit x^2, put $\cos\theta = 1$, $\sin\theta = \theta$. The process of differentiation will not then be very long, for it is easy to see beforehand what terms will disappear when we equate the differential coefficients (x', y', θ') to zero, and put for (x, y, θ) their initial values $(0, l\cos\beta, 0)$. We get

$$y_0''\cos\beta = \sin\beta\,(x_0'' + l\cos\beta\theta_0'').$$

This equation may also be obtained by an artifice which is often useful. The motion of G is made up of the motion of C and the motion of G relatively to C. Since C begins to describe a circle from rest, its acceleration along CO is zero. Again, the acceleration of G relatively to C when resolved along CO is $GC\dfrac{d^2\theta}{dt^2}\cos\beta$. The resolved acceleration of G is the sum of these two, but it is also

$$y_0''\cos\beta - x_0''\sin\beta.$$

Hence the equation follows at once.

In this case we require the differential equations only in their initial form. These are

$$\left.\begin{aligned} m{x_0}'' &= R_0 \sin\beta \\ m{y_0}'' &= mg - R_0\cos\beta \\ mk^2{\theta_0}'' &= R_0\, l \sin\beta\cos\beta \end{aligned}\right\},$$

where m is the mass of the body. Substituting in the geometrical equation we find

$$R_0 = mg \, . \, \frac{\cos\beta}{1 + \frac{l^2}{k^2}\sin^2\beta\cos^2\beta}\, .$$

The tension of any string, *before* the string OA was cut, may be found by the rules of statics, and is clearly $T_1 = \frac{mg}{3\cos\gamma}$, where γ is the angle AOG. Hence the change of tension can be found.

On Relative Motion or Moving Axes.

90. In many dynamical problems the relative motion of the different bodies of the system is frequently all that is required. In these cases it will be an advantage if we can determine this without finding the absolute motion of each body in space. Let us suppose that the motion relative to some one body (A) is required. There are then two cases to be considered, (1) when the body (A) has a motion of translation only, and (2) when it has a motion of rotation only. The case in which the body (A) has a motion both of translation and rotation may be regarded as a combination of these two cases. Let us consider these in order.

91. Let it be required to find the motion of any dynamical system relative to some moving point C. We may clearly reduce C to rest by applying to every element of the system an acceleration equal and opposite to that of C. It will also be necessary to suppose that an *initial* velocity equal and opposite to that of C has been applied to each element.

Let f be the acceleration of C at any time t. If every particle m of a body be acted on by the same accelerating force f parallel to any given direction, it is clear that these are together equivalent to a force $f\Sigma m$ acting at the centre of gravity. Hence to reduce any point C of a system to rest, it will be sufficient to apply to the centre of gravity of each body in a direction opposite to that of the acceleration of C a force measured by Mf, where M is the mass of the body and f the acceleration of C.

The point C may now be taken as the origin of co-ordinates. We may also take moments about it as if it were a point fixed in space.

Let us consider the equation of moments a little more minutely. Let (r, θ) be the polar co-ordinates of any element of a body whose mass is m referred to C as origin. The accelerations of the particle are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ and } \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right),$$

along and perpendicular to the radius vector r. Taking moments about C, we get

$$\Sigma m \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \left\{\begin{array}{l}\text{moment round } C \text{ of the impressed forces}\\ \text{plus the moment round } C \text{ of the reversed}\\ \text{effective forces of } C \text{ supposed to act at}\\ \text{the centre of gravity.}\end{array}\right.$$

If the point C be fixed in the body and move with it, $\frac{d\theta}{dt}$ will be the same for every element of the body, and, as in Art. 51, we have

$$\Sigma m \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = Mk^2\frac{d^2\theta}{dt^2}.$$

92. From the general equation of moments about a moving point we learn that we may use the equation

$$\frac{d\omega}{dt} = \frac{\text{moment of forces about } C}{\text{moment of inertia about } C}$$

in the following cases.

First. If the point C be fixed both in the body and in space; for then the acceleration of C is nothing.

Secondly. If the point C being fixed in the body move in space with uniform velocity; for the same reason as before.

Thirdly. If the point C be the centre of gravity; for in that case, though the acceleration of C is not zero, yet the moment vanishes.

Fourthly. If the point C be the instantaneous centre of rotation*, and the motion be a small oscillation. At the time t the body is turning about C, and the velocity of C is therefore zero. At the time $t+dt$, the body is turning about some point C' very near to C. Let $CC'=d\sigma$, then the velocity of C is $\omega d\sigma$. Hence in the time dt the velocity of C has increased from zero to $\omega d\sigma$, therefore its acceleration is $\omega \frac{d\sigma}{dt}$. To obtain the accurate equation of moments about C we must apply the effective force $\Sigma m \,.\, \omega \frac{d\sigma}{dt}$ in the reversed direction at the centre of gravity. But in small oscillations ω and $\frac{d\sigma}{dt}$ are both small quantities whose squares and products are to be neglected. Hence the moment of this force must be neglected, and the equation of motion will be the same as if C had been a fixed point.

It is to be observed that we may take moments about any point very near to the instantaneous centre of rotation, but it will usually be most convenient to take moments about the centre in its disturbed position. If there be any unknown reactions at the centre of rotation, their moments will then be zero.

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* If a body be in motion in one plane it is known that the actual displacement of every particle in the time dt is the same as if the body had been turned through some angle ωdt about some fixed point C. This may be proved in the same way as the corresponding proposition in Three Dimensions is proved in the next Chapter. See Art. 97. The point C is called the instantaneous centre of rotation, and ω is called the instantaneous angular velocity. See also Salmon's *Higher Plane Curves*, 1852, Arts. 246 and 264.

If the accurate equation of moments about the instantaneous centre be required, the value of $\omega \frac{d\sigma}{dt}$ must be found from the peculiar circumstances of the problem under consideration. For example, if a body roll on a curve, then the arc $d\sigma$ is described by C when the body has turned through an angle $\frac{d\sigma}{\rho} + \frac{d\sigma}{\rho'}$ where ρ, ρ' are the radii of curvature of the body and curve at the point of contact, the curvatures being supposed to be in opposite directions. Hence, since ωdt is the angle turned through by the body in the time dt,

$$\frac{d\sigma}{dt} = \frac{\omega}{\frac{1}{\rho} + \frac{1}{\rho'}}.$$

93. Example. *Two heavy particles whose masses are* m *and* m′ *are connected by an inextensible string, which is laid over the vertex of a double inclined plane whose mass is* M, *and which is capable of moving freely on a smooth horizontal plane. Find the force which must act on the wedge that the system may be in a state of relative equilibrium.*

Here it will be convenient to reduce the wedge to rest by applying to every particle an acceleration f equal and opposite to that of the wedge. Supposing this done the whole system is in equilibrium. If F be the required force, we have by resolving horizontally $(M + m + m')f = F$.

Let α, α' be the inclinations of the sides of the wedge to the horizontal. The particle m is acted on by mg vertically and mf horizontally. Hence the tension of the string is $m(g \sin\alpha + f\cos\alpha)$. By considering the particle m', we find the tension to be also $m'(g\sin\alpha' - f\cos\alpha')$. Equating these two we have

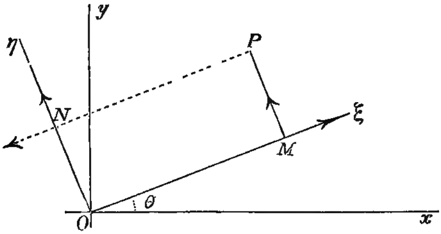
$$f = \frac{m'\sin\alpha' - m\sin\alpha}{m'\cos\alpha' + m\cos\alpha}\, g.$$

Hence F is found.

94. Next, let us consider the case in which we wish to refer the motion to two straight lines $O\xi$, $O\eta$ turning round a fixed origin O with angular velocity ω.

Let Ox, Oy be any fixed axes and let the angle $xO\xi = \theta$. Let $\xi = OM$, $\eta = PM$ be the co-ordinates of any point P.

It is evident that the motion of P is made up of the motions of the two points M, N by simple addition. The resolved parts of the velocity of M are $\frac{d\xi}{dt}$ and $\xi\omega$ along and perpendicular to OM. The resolved parts of the velocity of N are in the same way $\frac{d\eta}{dt}$ and $\eta\omega$ along and perpendicular to ON. By adding these with their proper signs we have



$$\left.\begin{array}{l}\text{velocity of } P\\ \text{parallel to } O\xi\end{array}\right\} = \frac{d\xi}{dt} - \eta\omega,$$

$$\left.\begin{array}{l}\text{velocity of } P\\ \text{parallel to } O\eta\end{array}\right\} = \frac{d\eta}{dt} + \xi\omega.$$

In the same way by adding the accelerations of M and N we have

$$\left.\begin{array}{r}\text{acceleration of } P\\ \text{parallel to } O\xi\end{array}\right\} = \frac{d^2\xi}{dt^2} - \xi\omega^2 - \frac{1}{\eta}\frac{d}{dt}(\eta^2\omega),$$

$$\left.\begin{array}{r}\text{acceleration of } P\\ \text{parallel to } O\eta\end{array}\right\} = \frac{d^2\eta}{dt^2} - \eta\omega^2 + \frac{1}{\xi}\frac{d}{dt}(\xi^2\omega).$$

By using these formulæ instead of $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ we may refer the motion to the moving axes $O\xi$, $O\eta$.

In a similar manner we may use polar co-ordinates. In this case if (r, ϕ) be the polar co-ordinates of P, we have

$$\left.\begin{array}{r}\text{acceleration of } P\\ \text{along rad. vect.}\end{array}\right\} = \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt} + \omega\right)^2,$$

$$\left.\begin{array}{r}\text{acceleration of } P\\ \text{perp. to rad. vect.}\end{array}\right\} = \frac{1}{r}\frac{d}{dt}\left\{r^2\left(\frac{d\phi}{dt} + \omega\right)\right\}.$$

95. EXAMPLE. *A particle under the action of any forces moves on a smooth curve which is constrained to turn with angular velocity ω about a fixed axis. Find the motion relative to the curve.*

Let us suppose the motion to be in three dimensions. Take the axis of Z as the fixed axis, and let the axes of ξ, η be fixed relatively to the curve. Then the equations of motion are

$$\left.\begin{array}{r}\dfrac{d^2\xi}{dt^2} - \xi\omega^2 - \dfrac{1}{\eta}\dfrac{d}{dt}(\eta^2\omega) = X + Rl\\[2ex] \dfrac{d^2\eta}{dt^2} - \eta\omega^2 + \dfrac{1}{\xi}\dfrac{d}{dt}(\xi^2\omega) = Y + Rm\\[2ex] \dfrac{d^2z}{dt} = Z + Rn\end{array}\right\} \ldots\ldots\ldots\ldots (1),$$

where X, Y, Z are the resolved parts of the impressed accelerating forces resolved parallel to the axes, R is the pressure on the curve, and (l, m, n) the direction-cosines of the direction of R. Then since R acts perpendicularly to the curve

$$l\frac{d\xi}{ds} + m\frac{d\eta}{ds} + n\frac{d\zeta}{ds} = 0.$$

Suppose the moving curve to be projected orthogonally on the plane of ξ, η, let σ be the arc of the projection, and $v' = \frac{d\sigma}{dt}$ be the resolved part of the velocity parallel to the plane of projection. Then the equations may be written in the form

$$\frac{d^2\xi}{dt^2} = X + \omega^2\xi + \frac{d\omega}{dt}\eta + 2\omega v'\frac{d\eta}{d\sigma} + Rl,$$

$$\frac{d^2\eta}{dt^2} = Y + \omega^2\eta - \frac{d\omega}{dt}\xi - 2\omega v'\frac{d\xi}{d\sigma} + Rm,$$

$$\frac{d^2z}{dt^2} = Z + Rn.$$

The two terms $2\omega v'\frac{d\eta}{d\sigma}$ and $-2\omega v'\frac{d\xi}{d\sigma}$ may be regarded as the resolved parts of a force $2\omega v'$ acting in a direction whose direction-cosines are

$$l' = \frac{d\eta}{d\sigma}, \quad m' = \frac{-d\xi}{d\sigma}, \quad n' = 0.$$

These satisfy the equation

$$l'\frac{d\xi}{ds} + m'\frac{d\eta}{ds} + n'\frac{d\zeta}{ds} = 0.$$

Hence the force is perpendicular to the tangent to the curve, and also perpendicular to the axis of rotation. Let R' be the resultant of the reaction R and of the force $2\omega v'$. Then R' also acts perpendicularly to the tangent, let (l'', m'', n'') be the direction-cosines of its direction.

The equations of motion therefore become

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} &= X + \omega^2\xi + \frac{d\omega}{dt}\eta + R'l'' \\ \frac{d^2\eta}{dt^2} &= Y + \omega^2\eta - \frac{d\omega}{dt}\xi + R'm'' \\ \frac{d^2z}{dt^2} &= Z + R'n''.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

These are the equations of motion of a particle moving on a *fixed curve*, and acted on in addition to the impressed forces by two extra forces, viz. (1) a force ω^2r tending directly from the axis, where r is the distance of the particle from the axis, and (2) a force $\frac{d\omega}{dt}r$ perpendicular to the plane containing the particle and the axis, and tending opposite to the direction of rotation of the curve.

In any particular problem we may therefore treat the curve as fixed. Thus suppose the curve to be turning round the axis with uniform angular velocity.

Then resolving along the tangent we have

$$v\frac{dv}{ds} = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} + \omega^2 r\frac{dr}{ds},$$

where r is the distance of the particle from the axis. Let V be the initial value of v, r_0 that of r. Then

$$v^2 - V^2 = 2\int(X\,dx + Y\,dy + Z\,dz) + \omega^2(r^2 - r_0^2).$$

Let v_0 be the velocity the particle would have had under the action of the same forces if the curve had been fixed. Then

$$v_0^2 - V^2 = 2\int(X\,dx + Y\,dy + Z\,dz).$$

Hence $$v^2 - v_0^2 = \omega^2(r^2 - r_0^2).$$

The pressure on the moving curve is not equal to the pressure on the fixed curve. The pressure R on the moving curve is clearly the resultant of the pressure R' on the fixed curve, and a pressure $2\omega v$ acting perpendicular both to the curve and to the axis in the direction of motion of the curve.

Thus suppose the curve to be plane and revolving uniformly about an axis perpendicular to its plane, and that there are no impressed forces. We have, resolving along the normal,

$$\frac{v^2}{\rho} = -\omega^2 r\sin\phi + R',$$

where ϕ is the angle r makes with the tangent.

If p be the perpendicular drawn from the axis on the tangent, we have, therefore,

$$R = \frac{v^2}{\rho} + \omega^2 p + 2\omega v.$$

If ω be variable, we have in a similar manner

$$R = \frac{v^2}{\rho} + \omega^2 p + 2\omega v + \frac{d\omega}{dt}\sqrt{r^2 - p^2}.$$

EXAMPLES.

1. A smooth sphere is placed on a horizontal plane, and another sphere resting on it is just disturbed from its position of unstable equilibrium. The two spheres being homogeneous, prove that whatever be their radii or weights, the upper sphere will leave the lower before the angle with the vertical made by the straight line joining the centres becomes equal to $\cos^{-1}\frac{2}{3}$.

2. A beam is rotating on a smooth horizontal plane about one extremity, which is fixed, under the action of no forces except the resistance of the atmosphere. Supposing the retarding effect of the resistance on a small element of the beam of length α to be $A\alpha\,(\text{vel.})^2$, then the angular velocity at the time t is given by

$$\frac{1}{\omega} - \frac{1}{\Omega} = \frac{Aa^4}{4M\kappa^2}t.$$

3. A circular hoop, which is free to move on a smooth horizontal plane, carries on it a small ring $\frac{1}{n}$th of its weight, the coefficient of friction between the two being μ. Initially the hoop is at rest and the ring has an angular velocity ω about the centre of the hoop. Show that the ring will be at rest on the hoop after a time $\frac{1+n}{\mu\omega}$.

4. A heavy circular wire has its plane vertical and its lowest point at a height h above a horizontal plane. A small ring is projected along the wire from its highest point with an angular velocity about its centre equal to $\pi n\sqrt{\frac{2g}{h}}$ at the

instant that the wire is let go. Show that when the wire reaches the horizontal plane, the particle will just have described n revolutions.

5. An inclined plane of mass M is capable of moving freely on a smooth horizontal plane. A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity. If x' be the horizontal space advanced by the inclined plane, x the part of the plane rolled over by the sphere, prove that

$$(M+m)\,x' = mx\cos\alpha,$$

$$\frac{7}{5}x - \cos\alpha x' = \frac{1}{2}g\sin\alpha t^2,$$

where α is the inclination of the plane to the horizon.

6. A heavy uniform sphere rolls on a rough plane and is acted on by a fixed centre of force in the plane varying inversely as the square of the distance; if the sphere be projected along the plane from a given point in it, in a direction opposite to that of the centre of force, find the roughness of the plane at any point, supposing the whole of it to be required.

7. Two equal uniform rods of length $2a$, loosely jointed at one extremity, are placed symmetrically upon a fixed smooth sphere of radius $\frac{a\sqrt{2}}{3}$, and raised into a horizontal position so that the hinge is in contact with the sphere. If they be allowed to descend under the action of gravity, show that, when they are first at rest, they are inclined at an angle $\cos^{-1}\frac{1}{3}$ to the horizon, that the points of contact with the sphere are the centres of oscillation of the rods relatively to the hinge, that the pressure on the sphere at each point of contact equals one-fourth the weight of either rod, and that there is no strain on the hinge.

8. Two circular discs are on a smooth horizontal plane; one, whose radius is n times that of the other, is fixed: an elastic string wraps round them so that those portions of it not in contact with the discs are common interior tangents,

the natural length of the string being the sum of the circumferences. The moveable disc is drawn from the other till the tension of the string is T, prove that if it be now let go, the velocity acquired when it comes in contact with the fixed disc will be

$$\frac{T}{\lambda}\sqrt{\frac{2(n+1)\pi a.\lambda}{m}},$$

where m is the mass of the moving disc, λ the modulus of elasticity, a the radius of the moving disc.

9. Two straight equal and uniform rods are connected at their ends by two strings of equal length a, so as to form a parallelogram. One rod is supported at its centre by a fixed axis about which it can turn freely, this axis being perpendicular to the plane of motion which is vertical. Show that the middle point of the lower rod will oscillate in the same way as a simple pendulum of length a, and that the *angular* motion of the rods is independent of this oscillation.

10. A fine string is attached to two points A, B in the same horizontal plane, and carries a weight W at its middle point. A rod whose length is AB and weight W, has a ring at either end, through which the string passes, and is let fall from the position AB. Show that the string must be at least $\frac{5}{3}AB$, in order that the weight may ever reach the rod.

Also if the system be in equilibrium, and the weight be slightly and vertically displaced, the time of its small oscillations is $2\pi\sqrt{\frac{AB}{3g\sqrt{3}}}$.

11. A fine thread is enclosed in a smooth circular tube which rotates freely about a vertical diameter; prove that, in the position of relative equilibrium, the inclination (θ) to the vertical, of the diameter through the centre of gravity of the thread, will be given by the equation

$$\cos\theta = \frac{g}{a\omega^2\cos\beta},$$

where ω is the angular velocity of the tube, a its radius, and $2a\beta$ the length of the thread. Explain the case in which the value of $a\omega^2 \cos\beta$ lies between g and $-g$.

12. A smooth wire without inertia is bent into the form of a helix which is capable of revolving about a vertical axis coinciding with a generating line of the cylinder on which it is traced. A small heavy ring slides down the helix, starting from a point in which this vertical axis meets the helix: prove that the angular velocity of the helix will be a maximum when it has turned through an angle θ given by the equation

$$\cos^2\theta + \tan^2\alpha + \theta \sin 2\theta = 0,$$

α being the inclination of the helix to the horizon.

13. A spherical shell (of radius a and mass m) rolls along a rough horizontal plane, whilst a smooth particle P oscillates within the shell in the vertical plane in which the centre of the shell moves, the particle being never very far from the lowest point. Show that the time of its oscillation will be the same as that of a simple pendulum of length

$$\frac{ma(a^2+k^2)}{m(a^2+k^2)+Pa^2},$$

where k is the radius of gyration of the shell about a diameter.

14. A spherical hollow of radius a is made in a cube of glass of mass M, and a particle of mass m is placed within. The cube is then set in motion on a smooth horizontal plane so that the particle just gets round the sphere, remaining in contact with it. If the velocity of projection be V, prove that

$$V^2 = 5ag + 4ag\frac{m}{M}.$$

15. A perfectly rough ball is placed within a hollow cylindrical garden-roller at its lowest point, and the roller is then drawn along a level walk with a uniform velocity V. Show that the ball will roll quite round the interior of the roller, if V^2 be $> \frac{27}{7} g(b-a)$, a being the radius of the ball, and b of the roller.

16. A semicircular wire AB of radius a is rotating on a smooth horizontal plane about one extremity A with a constant angular velocity ω. If $a\phi$ be the arc between the fixed point A and the point where the tendency to break is greatest, prove that $\tan\phi = \pi - \phi$. If the extremity B be suddenly fixed and the extremity A let go, prove that the tendency to break is greatest at a point P where

$$\frac{1}{2}\tan PBA = PBA.$$

17. Two of the angles of a heavy square lamina, a side of which is a, are connected with two points equally distant from the centre of a rod of length $2a$, so that the square can rotate about the rod. The weight of the square is equal to the weight of the rod, and when supported by its extremities in a horizontal position is on the point of breaking. The rod is then held by its extremities in a vertical position, and an angular velocity ω is then impressed on the square. Show that it will break if $\omega > \sqrt{\frac{3g}{a}}$.

18. Three equal rods placed in a straight line are jointed by hinges to one another, they move with a velocity v perpendicular to their lengths; if the middle point of the middle one become suddenly fixed, show that the extremities of the other two will meet in a time $\frac{4\pi a}{9v}$, a being the length of each rod.

19. AB, BC are two equal uniform rods loosely jointed at B, and moving with the same velocity in a direction perpendicular to their length; if the end A be suddenly fixed, show that the initial angular velocity of AB is three times that of BC. Also show that in the subsequent motion of the rods, the greatest angle between them equals $\cos^{-1}\frac{2}{3}$, and that when they are next in a straight line, the angular velocity of BC is nine times that of AB.

20. Three equal heavy uniform beams jointed together are laid in the same right line on a smooth table, and a given

horizontal impulse is applied at the middle point of the centre beam in a direction perpendicular to its length; show that the instantaneous impulse on each of the other beams is one-sixth of the given impulse.

21. Three beams of like substance, joined together so as to form one beam, are laid on a smooth horizontal table. The two extreme beams are equal in length, and one of them receives a blow at its free extremity in a direction perpendicular to its length. Determine the length of the middle beam in order that the greatest possible angular velocity may be given to the third.

Result. The length of the middle beam must be to either of the outer beams as $\sqrt{3} : 2$.

22. Two rough rods A, B are placed parallel to each other and in the same horizontal plane. Another rough rod C is laid across them at right angles, its centre of gravity being half way between them. If C be raised through any angle α and let fall, determine the conditions that it may oscillate, and show that if its length be equal to twice the distance between A and B, the angle θ through which it will rise in the n^{th} oscillation is given by the equation

$$\sin\theta = \left(\frac{1}{7}\right)^{2n} . \sin\alpha.$$

23. A rod moveable in a vertical plane about a hinge at its upper end has a given uniform rod attached to its lower end by a hinge about which it can turn freely in the same vertical plane as the upper rod; at what point must the lower rod be struck horizontally in that same vertical plane that the upper rod may initially be unaffected by the blow?

24. A uniform beam is balanced about a horizontal axis through its centre of gravity, and a perfectly elastic ball is let fall from a height h on one extremity; determine the motion of the beam and ball.

Result. Let M, m be the masses of beam and ball, $2a$ = length of beam, V, V' the velocities of ball at the moments before and after impact, ω' the angular velocity of the beam. Then

$$\omega' = \frac{6mV}{(M+3m)\,a}, \quad V' = V \cdot \frac{3m-M}{3m+M}.$$

25. A free lamina of any form is turning in its own plane about an instantaneous centre of rotation S and impinges on a fixed obstacle P, situated in the straight line joining the centre of gravity G to S. Find the position of P, *first*, that the centre of gravity may be reduced to rest, *secondly*, that its velocity after impact may be the same as before but reversed in direction.

Result. In the first case, P coincides either with G or with the centre of oscillation. In the second case the points $x = GP$ are found from the equation $x^2 - \frac{k^2}{2h}x + \frac{k^2}{2} = 0$, where $SG = h$.

26. A ball spinning about a vertical axis moves on a smooth table and impinges directly on a perfectly rough vertical cushion; show that the vis viva of the ball is diminished in the ratio $10^2 + 14\tan^2\theta : \frac{10}{e^2} + 49\tan^2\theta$, where e is the elasticity of the ball and θ the angle of reflexion.

27. A square is moving freely about a diagonal with angular velocity ω, when one of the angular points not in that diagonal becomes fixed; determine the impulsive pressure on the fixed point, and show that the instantaneous angular velocity will be $\frac{\omega}{7}$.

28. The points $ABCD$ are the angular points of a square; AB, CD are two equal similar rods connected by the string BC, equal in length to either of them. The point A receives an impulse in the direction AD, show that the initial velocity of A is seven times that of the point D.

29. A rhombus is formed of four rigid uniform rods, each of length $2a$, freely jointed at their extremities. If the rhombus be laid on a smooth horizontal table and a blow be

applied at right angles to any one of the rods, the rhombus will begin to move as a rigid body if the blow be applied at a point distant $a(1 - \cos\alpha)$ from an acute angle, where α is the acute angle.

30. A rectangle is formed of four uniform rods of lengths $2a$ and $2b$ respectively, which are connected by hinges at their ends. The rectangle is revolving about its centre on a smooth horizontal plane with an angular velocity n, when a point in one of the sides of length $2a$ suddenly becomes fixed. Show that the angular velocity of the sides of length $2b$ immediately becomes $\frac{3a+b}{6a+4b}n$. Find also the change in the angular velocity of the other sides and the impulsive action at the point which becomes fixed.

31. Equal beams AB, BC, CD...... are connected by hinges, the beams are placed on a smooth horizontal plane, each at right angles to the two adjacent, so as to form a figure resembling a set of steps, and an impulse is given at the end A along AB. If X_n be the impulsive action at the n^{th} angular point, show that $X_{2n+1} - 5X_{2n+2} - 2X_{2n+3} = 0$ and $X_{2n+2} - 5X_{2n+1} - 2X_{2n} = 0$. Thence find X_n.

32. Three equal uniform inelastic rods loosely jointed together are laid in a straight line on a smooth horizontal table, and the two outer ones are set in motion about the ends of the middle one with equal angular velocities (1) in the same direction and (2) in opposite directions. Prove that in the first case, when the outer rods make the greatest angle with the direction of the middle one produced on each side the common angular velocity of the three is $\frac{4\omega}{7}$, and in the second case after the impact of the two outer rods the triangle formed by them will move with uniform velocity $\frac{2a\omega}{3}$, $2a$ being the length of each rod.

33. An inelastic sphere of radius a sliding on a smooth horizontal plane impinges on a fixed rough point at a height

c above the plane, show that if the velocity of the sphere be $\sqrt{\frac{2gc(a^2+k^2)}{(a-c)^2}}$ it will just roll over the point.

34. An imperfectly elastic sphere sliding with velocity V on a smooth horizontal plane impinges on a fixed perfectly rough point, if the angle which the line joining the centre of the sphere and the point makes with the vertical be $\tan^{-1}\sqrt{\frac{5}{7e}}$, where e is the normal elasticity, then the centre of the sphere will come to rest in a time from the instant of impact equal to $2\sqrt{\frac{5e}{7}}\,\frac{V}{g(1-e)}$.

35. A heavy circular lamina of radius a and mass M rolls on the inside of a rough circular arc of twice its radius fixed in a vertical plane. Find the motion. If the lamina be placed at rest in contact with the lowest point, the impulse which must be applied horizontally that it may rise as high as possible (not going all round), without falling off, is $M\sqrt{3ag}$.

36. A string without weight is coiled round a rough horizontal cylinder, of which the mass is M and radius a, and which is capable of turning round its axis. To the free extremity of the string is attached a chain of which the mass is m and the length l; if the chain be gathered close up and then let go, prove that if θ be the angle through which the cylinder has turned after a time t before the chain is fully stretched,

$$Ma\theta = \frac{m}{l}\left(\frac{gt^2}{2} - a\theta\right)^2.$$

37. Two strings of equal length have each an extremity tied to a weight C and their other extremities tied to two points A, B in the same horizontal. If one be cut the tension of the other is instantaneously altered in the ratio $1 : 2\cos^2\frac{C}{2}$.

38. Two equal rods AC, BC, are freely connected at C, and hooked to A and B, two points in the same horizontal line, each rod being then inclined at an angle α to the horizon. The hook B suddenly giving way, prove that the direction of the strain at C is instantaneously shifted through an angle

$$\tan^{-1}\left(\frac{1+6\sin^2\alpha}{1+6\cos^2\alpha}\cdot\frac{2-3\cos^2\alpha}{3\sin\alpha\cos\alpha}\right).$$

39. An elliptic lamina is supported with its plane vertical and transverse axis horizontal by two weightless pegs passing through the foci. If one pin be released show that if the eccentricity of the ellipse be $\sqrt{\frac{2}{5}}$, the pressure on the other pin will be initially unaltered.

40. Two particles A, B are connected by a fine string; A rests on a rough horizontal table and B hangs vertically at a distance l below the edge of the table. If A be on the point of motion and B be projected horizontally with a velocity u, show that A will begin to move with acceleration $\frac{\mu}{\mu+1}\,\frac{u^2}{l}$, and that the initial radius of curvature of B's path will be $(\mu+1)\,l$, where μ is the coefficient of friction.

41. Three equal particles A, B, C repelling each other with any forces, are tied together by three strings of unequal length, so as to form a triangle right-angled at A. If the string joining B and C be cut, prove that the instantaneous changes of tension of the strings joining BA, CA are $\frac{1}{2}T\cos B$ and $\frac{1}{2}T\cos C$ respectively, where B and C are the angles opposite the strings joining CA, AB respectively.

42. Two particles (m, m') are connected by a string passing through a small fixed ring and are held so that the string is horizontal; their distances from the ring being a and a', they are let go. If ρ, ρ' be the initial radii of curvature of their paths, prove that

$$\frac{m}{\rho}=\frac{m'}{\rho'}, \quad \text{and } \frac{1}{\rho}+\frac{1}{\rho'}=\frac{1}{a}+\frac{1}{a'}.$$

43. A sphere whose centre of gravity is not in its centre is placed on a rough table; the coefficient of friction being μ, determine whether it will begin to slide or to roll.

44. An equilateral triangle is suspended from a point by three strings, each equal to one of the sides, attached to its angular points; if one string be cut, show that the tensions of the other two are diminished in the ratio of 36 : 43.

45. A horizontal rod of mass m and length $2a$, hangs by two parallel strings of length $2a$ attached to its ends: an angular velocity ω being suddenly communicated to it about a vertical axis through its centre, show that the initial increase of tension of either string equals $\frac{ma\omega^2}{4}$, and that the rod will rise through a space $\frac{a^2\omega^2}{6g}$.

46. A uniform solid, in the form of a paraboloid of revolution, rests with its vertex on a smooth horizontal plane. It is divided symmetrically by a vertical plane. Explain why the pressure on the plane is instantly diminished; find the change of pressure.

47. A circular ring is fixed in a vertical position upon a smooth horizontal plane, and a small ring is placed on the circle, and attached to the highest point by a string, which subtends an angle α at the centre; prove that if the string be cut and the circle left free, the pressures on the ring before and after the string is cut are in the ratio $M + m\sin^2\alpha : M\cos\alpha$, m and M being the masses of the ring and circle.

48. Two uniform equal rods, each of mass m, are placed in the form of the letter X on a smooth horizontal plane, the upper and lower extremities being connected by equal strings; show that whichever string be cut, the tension of the other is the same function of the inclination of the rods, and initially is $\frac{3}{8}mg\sin\alpha$, where α is the initial inclination of the rods.

49. One extremity C of a rod is made to revolve with uniform angular velocity n in the circumference of a circle of radius a, while the rod itself is made to revolve in the opposite direction with the same angular velocity about that extremity. The rod initially coincides with a diameter, and a smooth ring capable of sliding freely along the rod is placed at the centre of the circle. If r be the distance of the ring from C at the time t, prove

$$r = \frac{2a}{5}(e^{nt} + e^{-nt}) + \frac{a}{5}\cos 2nt.$$

CHAPTER V.

MOTION OF A RIGID BODY IN THREE DIMENSIONS.

The Geometry of the Motion of a Rigid Body.

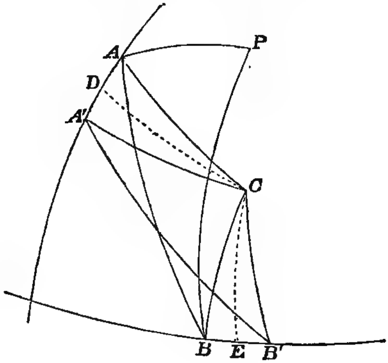
96. If the particles of a body be rigidly connected, then whatever be the nature of the motion generated by the forces, there must be some general relations between the motions of the particles of the body. These must be such that if the motion of three points not in the same straight line be known, that of every other point may be deduced. It will then in the first place be our object to consider the general character of the motion of a rigid body apart from the forces that produce it, and to reduce the determination of the motion of every particle to as few independent quantities as possible: and in the second place we shall consider how when the forces are given these independent quantities may be found.

97. *One point of a moving rigid body being fixed, it is required to deduce the general relations between the motions of the other points of the body.*

Let O be the fixed point and let it be taken as the centre of a moveable sphere which we shall suppose fixed in the body. Let the radius vector to any point Q of the body cut the sphere in P, then the motion of every point Q of the body will be represented by that of P.

If the displacements of two points A, B, on the sphere in the small time dt be given as AA', BB', then clearly the displacement of any other point P on the sphere may be found by constructing on $A'B'$ as base a triangle $A'P'B'$ similar and equal to APB. Then PP' will represent the displacement of P. It may be assumed as evident, or it may be proved as in Euclid, that on the same base and on the same side of it there cannot be two triangles on the same sphere, which have their sides terminated in one extremity of the base equal to one another, and likewise those terminated in the other extremity.

Let D and E be the middle points of the arcs AA', BB', and let DC, EC be arcs of great circles drawn perpendicular to AA', BB' respectively. Then clearly $CA = CA'$ and $CB = CB'$, and therefore since the bases AB, $A'B'$ are equal, the two triangles ACB, $A'CB'$ are equal and similar. Hence the displacement of C is zero.



Also it is evident since the displacements of O and C are zero, that the displacement of every point in the straight line OC is also zero.

Hence if a body be in motion in any manner about a fixed point O, *there is at every instant a straight line* OC *such that the displacement of every point in it in the time* dt *is zero.*

Since every point in OC is at rest, we may suppose this straight line to be fixed in space. The body may therefore be brought from any position AB into another $A'B'$ by a rotation about OC as an axis through an angle PCP' such that any one point P is brought into coincidence with its new position P'. Then every point of the body will be brought from its first to its second position. Let this angle be called $d\theta$.

98. **Def.** The ultimate ratio of this angle $d\theta$ to the time dt is called the angular velocity of the body about OC, and the straight line OC is called the instantaneous axis at the time t.

The angular velocity may also be defined to be the angle through which the body would turn in a unit of time if it continued to turn throughout that unit with the same angular velocity which it had at the proposed instant, and about the same axis.

99. *To explain what is meant by a body having angular velocities about more than one axis at the same time.*

A body in motion is said to have an angular velocity ω about a straight line, when, the body being turned round this straight line through an angle ωdt, every point of the body is brought from its position at the time t to its position at the time $t + dt$.

Suppose that during three successive intervals each of time dt, the body is turned successively round three different straight lines OA, OB, OC meeting at a point O through angles $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$. Then we shall first prove that the final position is the same in whatever order these rotations are effected. Let P be any point in the body, and let its distances from OA, OB, OC, respectively be r_1, r_2, r_3. First let the body be turned round OA, then P receives a displacement $\omega_1 r_1 dt$. By this motion let r_2 be increased to $r_2 + dr_2$, then the displacement caused by the rotation about OB will be in magnitude $\omega_2 (r_2 + dr_2)\, dt$. But according to the principles of the Differential Calculus we may in the limit neglect the quantities of the second order, and the displacement becomes $\omega_2 r_2 dt$. So also the displacement due to the remaining rotation will be $\omega_3 r_3 dt$. And these three results will be the same in whatever order the rotations take place. In a similar manner we can prove that the *directions* of these displacements will be independent of the order. The final displacement is the diagonal of the parallelopiped described on these three lines as sides, and is therefore independent of the order of the rotations. Since then the three rotations are quite independent, they may be said to take place simultaneously.

When a body is said to have angular velocities about three different axes it is only meant that the motion may be determined as follows. Divide the whole time into a number of small intervals each equal to dt. During each of these, turn the body round the three axes successively, through angles $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$. Then when dt diminishes without limit the motion during the whole time will be accurately represented.

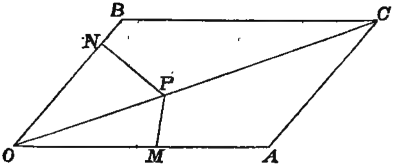
100. *If two angular velocities about two axes* OA, OB *be represented in magnitude and direction by the two lengths* OA, OB; *then the diagonal* OC *of the parallelogram constructed on* OA, OB *as sides will be the resultant axis of rotation, and its length will represent the magnitude of the resultant angular velocity.* This Prop. is usually called "The parallelogram of angular velocities."

Let P be any point in OC, and let PM, PN be drawn perpendicular to OA, OB. Since OA represents the angular velocity about OA and PM is the perpendicular distance of P from OA, the product $OA . PM$ will represent the velocity of P due to the angular velocity about OA. Similarly $OB . PN$ will represent the velocity of P due to the angular velocity about OB. Since P is on the left-hand side of OA and on the right-hand side of OB, as we respectively look along these directions, it is evident that these velocities are in opposite directions.

Hence the velocity of any point P is represented by

$$\begin{aligned} & OA . PM - OB . PN \\ & = OP\{OA . \sin COA - OB . \sin COB\} \\ & = 0. \end{aligned}$$

Therefore the point P is at rest and OC is the resultant axis of rotation.



Let ω be the angular velocity about OC, then the velocity of any point A in OA is perpendicular to the plane AOB and is represented by ω. perpendicular distance of A from $OC = \omega \,.\, OA \sin COA$. But since the motion is also determined by the two given angular velocities about OA, OB, the motion of the point A is also represented by OB. perpendicular distance of A from $OB = OB \,.\, OA \sin BOA$;

$$\therefore \omega = OB \,.\, \frac{\sin BOA}{\sin COA} = OC.$$

Hence the angular velocity about OC is represented in magnitude by OC.

From this proposition we may deduce as a corollary "the parallelogram of angular accelerations." For if OA, OB represent the additional angular velocities impressed on a body at any instant, it follows that the diagonal OC will represent the resultant additional angular velocity in direction and magnitude.

101. This proposition shows that angular velocities and angular accelerations may be compounded and resolved by the same rules and in the same way as if they were forces. Thus an angular velocity ω about any given axis may be resolved into two, $\omega \cos \alpha$ and $\omega \sin \alpha$, about axes at right angles to each other and making angles α and $\frac{\pi}{2} - \alpha$ with the given axis.

If a body have angular velocities $\omega_1, \omega_2, \omega_3$ about three axes Ox, Oy, Oz at right angles, they are together equivalent to a single angular velocity ω, where $\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$, about an axis making angles with the given axes whose cosines are respectively $\frac{\omega_1}{\omega}, \frac{\omega_2}{\omega}, \frac{\omega_3}{\omega}$. This may be proved, as in the corresponding proposition in Statics, by compounding the three angular velocities, taking them two at a time.

It will however be needless to recapitulate the several propositions proved for forces in Statics with special reference to angular velocities. We may use "the triangle of angular velocities" or the other rules for compounding several angular velocities together, without any further demonstration.

102. *A body has angular velocities ω, ω' about two parallel axes* OA, O′B *distant* a *from each other, to find the resulting motion.*

Since parallel straight lines may be regarded as the limit of two straight lines which intersect at a very great distance, it follows from the parallelogram of angular velocities that the two given angular velocities are equivalent to an angular velocity about some parallel axis $O''C$ lying in the plane containing OA, $O'B$.

Let x be the distance of this axis from OA, and suppose it to be on the same side of OA as OB. Let Ω be the angular velocity about it.

Consider any point P, distant y from OA and lying in the plane of the three axes. The velocity of P due to the rotation about OA is ωy, the velocity due to the rotation about $O'B$ is $\omega'(y-a)$. But these two together must be equivalent to the velocity due to the resultant angular velocity Ω about $O''C$, and this is $\Omega(y-x)$,

$$\therefore\ \omega y + \omega'(y-a) = \Omega(y-x).$$

This equation is true for all values of y,

$$\left.\begin{aligned} \therefore\ \Omega &= \omega + \omega' \\ x &= \frac{a\omega'}{\Omega} \end{aligned}\right\}.$$

This is the same result we should have obtained if we had been seeking the resultant of two *forces* ω, ω' acting along OA, OB.

If $\omega = -\omega'$, the resultant angular velocity vanishes, but x is infinite. The velocity of any point P is in this case

$$\omega y + \omega' (y - a) = a\omega,$$

which is independent of the position of P.

The result is that two angular velocities, each equal to ω but tending to turn the body in opposite directions about two parallel axes at a distance a from each other, are equivalent to a *linear velocity* represented by $a\omega$. This corresponds to the proposition in Statics that "a couple" is properly measured by its moment.

We may deduce as a corollary, that a motion of rotation ω about an axis OA is equivalent to an equal motion of rotation about a parallel axis $O'B$ plus a motion of translation $a\omega$ perpendicular to the plane containing OA, $O'B$, and in the direction in which $O'B$ moves.

103. *Every motion of a rigid body may be represented by a combination of the two following motions.*

First. A motion of translation, whereby every particle is moved parallel to the direction of motion of any assumed point rigidly connected with the body, and with the same velocity.

Secondly. A motion of rotation of the whole body about some axis through this assumed point.

It is evident that the change of position of the body can be effected *first*, by moving any point O from its old to its new position by a motion of translation, and *secondly*, retaining this point O fixed, by moving any two points of the body not in one straight line with O into their new positions. This last motion has been proved to be equivalent to a rotation about a single axis through O.

Let ω_1, ω_2, ω_3 be the resolved parts of the angular velocity about any three rectangular axes meeting at O; u, v, w the linear velocities of O parallel to these axes. Then these six quantities determine the state of motion of the body at the instant under consideration. It is obvious that we may choose any point of the body for the point O, so that the motion may be represented in a great many different ways. The relations which exist between these may be inferred from the corresponding propositions in Statics.

104. *To explain a certain analogy which exists between Statics and Dynamics.*

All propositions in Statics relating to the composition and resolution of forces and couples are founded on these theorems:

1. The parallelogram of forces and the parallelogram of couples.

2. A force F is equivalent to any equal and parallel force together with a couple Fp, where p is the distance between the forces.

Corresponding to these we have in Dynamics the following theorems on the motion of a rigid body:

1. The parallelogram of angular velocities and the parallelogram of linear velocities.

2. An angular velocity ω is equivalent to an equal angular velocity about a parallel axis together with a linear velocity equal to ωp, where p is the distance between the parallel axes.

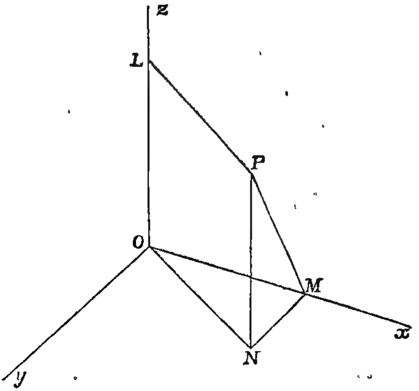
It follows that every proposition in Statics relating to forces has a corresponding proposition in Dynamics relating to the motion of a rigid body, and these two may be proved in the same way.

It is proved in Statics that a system of forces and couples is generally equivalent to a single force and a single couple, and that these may be reduced to a resultant R acting along a line called the central axis, and a couple G about that axis. Or they may also be reduced to a resultant R of the same magnitude as before, acting along any line parallel to the central axis at any chosen distance c from it, together with a couple G' about an axis perpendicular to the line whose length is c, and inclined to the resultant R at an angle θ. Then we know that $G' = \sqrt{G^2 + R^2c^2}$, and is a minimum when $c = 0$, and also that $\tan\theta = \frac{Rc}{G}$.

The same train of reasoning by which these results were established, will establish the following proposition. The general motion of a body having been reduced to a motion of translation and one of rotation, these are equivalent to a motion of rotation ω about a line called the central axis, and a translation V along that axis. Or they may also be reduced to a rotation ω of the same magnitude as before about any line parallel to the central axis, and at any chosen distance c from it, together with a translation V' along a line perpendicular to the line c, and inclined to the axis of ω at an angle θ. Then we know that $V' = \sqrt{V^2 + c^2\omega^2}$, and is a minimum when $c = 0$, and also that $\tan\theta = \frac{c\omega}{V}$.

In a similar manner many other propositions may be established.

105. *Given the angular velocities* $\omega_1, \omega_2, \omega_3$ *of a body about three axes* Ox, Oy, Oz *at right angles, to determine the actual velocities of a particle whose co-ordinates are* x, y, z.



These angular velocities are supposed positive when they tend the same way round the axes that positive couples tend in Statics. Thus the positive directions of ω_1, ω_2, ω_3 are respectively from y to z, from z to x, and from x to y.

Let us determine the velocity of P parallel to the axis of z. Let PN be the ordinate z, and let PM be drawn perpendicular to Ox. The velocity of P due to the rotation about Ox is clearly $\omega_1 PM$. Resolving this along NP we get

$$\omega_1 PM \sin NPM = \omega_1 y.$$

Similarly that due to the rotation about Oy is $-\omega_2 x$; and that due to the rotation about Oz is 0. Hence the whole velocity of P parallel to Oz is

$$\frac{dz}{dt} = \omega_1 y - \omega_2 x,$$

and the velocities parallel to the other axes are

$$\frac{dx}{dt} = \omega_2 z - \omega_3 y,$$

$$\frac{dy}{dt} = \omega_3 x - \omega_1 z.$$

The quantities ω_1, ω_2, ω_3 are called the angular velocities of the *body* about the axes of x, y, z respectively, but they must be carefully distinguished from the angular velocities of any particular *particle* of the body about the same axes. Let P be any particle of the body whose co-ordinates are x, y, z, and draw $PL = r$ perpendicular to the axis of z. Let θ be the angle xON, then the instantaneous angular velocity of P about Oz is $\frac{d\theta}{dt}$.

But $$r^2 \frac{d\theta}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt}$$

$$= \omega_3 r^2 - xz\omega_1 - yz\omega_2,$$

by substituting for $\frac{dx}{dt}$, $\frac{dy}{dt}$, their values just found;

$$\therefore \frac{d\theta}{dt} = \omega_3 - \omega_1 \frac{xz}{r^2} - \omega_2 \frac{yz}{r^2}.$$

Hence the angular velocity of a particle about Oz is the same as that of the body when the particle lies in the plane of xy, or when it lies in the plane given by

$$y = -x\frac{\omega_1}{\omega_2}.$$

If the axes be themselves moving in any manner, these equations only give the linear velocities of the particle relatively to the axes. Thus suppose the directions of the axes to be fixed in space, but the origin O to be in motion with a velocity V whose resolved parts parallel to the axes are respectively u, v, w. Then the formulæ will become

$$\left.\begin{aligned} \frac{dx}{dt} &= u + \omega_2 z - \omega_3 y \\ \frac{dy}{dt} &= v + \omega_3 x - \omega_1 z \\ \frac{dz}{dt} &= w + \omega_1 y - \omega_2 x \end{aligned}\right\}.$$

106. The equations of the last article will enable us to deduce without difficulty the results of Art. 104. Suppose the motion to be given, as before, by the linear velocities (u, v, w) of some point O and the angular velocities $(\omega_1, \omega_2, \omega_3)$.

Let the same motion be also represented by the linear velocities u', v', w' parallel to the axes, of some other point O' and by angular velocities $\omega_1', \omega_2', \omega_3'$ about axes parallel to the co-ordinate axes and meeting in O'. Let (ξ, η, ζ) be the co-ordinates of O'. We have now two representations of the same motion, both these must give the same result for the linear velocities of any point. Hence

$$\left.\begin{aligned} u + \omega_2 z - \omega_3 y &= u' + \omega_2'(z-\zeta) - \omega_3'(y-\eta) \\ v + \omega_3 x - \omega_1 z &= v' + \omega_3'(x-\xi) - \omega_1'(z-\zeta) \\ w + \omega_1 y - \omega_2 x &= w' + \omega_1'(y-\eta) - \omega_2'(x-\xi) \end{aligned}\right\} \dots\dots (1),$$

must be true for all values of x, y, z.

This gives $\omega_1' = \omega_1$, $\omega_2' = \omega_2$, $\omega_3' = \omega_3$, so that whatever origin is chosen, the angular velocity is always the same in direction and magnitude.

Also (ξ, η, ζ) may be so chosen that the velocity of O' is along the axis of rotation; in this case we have (u', v', w') proportional to $(\omega_1, \omega_2, \omega_3)$. The equation to the locus of O' is therefore

$$\frac{u+\omega_2\zeta-\omega_3\eta}{\omega_1}=\frac{v+\omega_3\xi-\omega_1\zeta}{\omega_2}=\frac{w+\omega_1\eta-\omega_2\xi}{\omega_3} \ldots\ldots (2).$$

By multiplying the numerator and denominator of each of these fractions by ω_1, ω_2, ω_3 respectively, and adding them together, we see that each of them is

$$=\frac{u\omega_1+v\omega_2+w\omega_3}{\omega^2}.$$

The motion of the body is thus represented by a motion of translation along the straight line whose equations are (2) and an angular velocity equal to ω about it. This straight line is called the central axis.

If the motion be such that $u\omega_1+v\omega_2+w\omega_3=0$, and ω_1, ω_2, ω_3 do not all vanish, each of the equalities in (2) is zero, and hence by equation (1) $u'=0$, $v'=0$, $w'=0$. The motion is therefore equivalent to a rotation about the central axis, without translation. This also is evident from the analogy explained in Art. 104.

107. *To determine the general equations of motion of a body about a fixed point.*

Let the fixed point O be taken as origin, and let x, y, z be the co-ordinates at time t of any particle m referred to any rectangular axes fixed in space. Let Xm, Ym, Zm be the impressed forces acting on this element, and let L, M, N be the moments of all these forces about the axes of co-ordinates, and let P, Q, R be the pressures of the fixed point on the body.

Then by D'Alembert's Principle, if the effective forces $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, $m\frac{d^2z}{dt^2}$ be applied to every particle m in a reversed direction, there will be equilibrium between these forces and the impressed forces. Taking moments therefore about the axes, we have

$$\left.\begin{aligned}\Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)&=L\\ \Sigma m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right)&=M\\ \Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)&=N\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots \text{(I).}$$

Resolving parallel to the axes, we have

$$\left.\begin{aligned}\Sigma m\frac{d^2x}{dt^2}&=P+\Sigma mX\\ \Sigma m\frac{d^2y}{dt^2}&=Q+\Sigma mY\\ \Sigma m\frac{d^2z}{dt^2}&=R+\Sigma mZ\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots \text{(II).}$$

To simplify these equations, let ω_x, ω_y, ω_z be the angular velocities about the axes. Then

$$\frac{dx}{dt}=\omega_y z-\omega_z y,$$

$$\frac{dy}{dt}=\omega_z x-\omega_x z,$$

$$\frac{dz}{dt}=\omega_x y-\omega_y x;$$

$$\therefore\ \frac{d^2x}{dt^2}=z\frac{d\omega_y}{dt}-y\frac{d\omega_z}{dt}+\omega_y(\omega_x y-\omega_y x)-\omega_z(\omega_z x-\omega_x z),$$

$$\frac{d^2y}{dt^2}=x\frac{d\omega_z}{dt}-z\frac{d\omega_x}{dt}+\omega_z(\omega_y z-\omega_z y)-\omega_x(\omega_x y-\omega_y x).$$

Substituting in the last of equations (I) we get

$$\left.\begin{aligned}\Sigma m\,(x^2+y^2)\,\frac{d\omega_z}{dt}-\Sigma myz\,.\,\frac{d\omega_y}{dt}-\Sigma mxz\,.\,\frac{d\omega_x}{dt}\\ -\Sigma mxy\,.\,(\omega_x^2-\omega_y^2)+\Sigma m\,(x^2-y^2)\,\omega_x\omega_y-\Sigma myz\,.\,\omega_x\omega_z\\ +\Sigma mxz\,.\,\omega_y\omega_z\end{aligned}\right\}=N.$$

The other two equations may be treated in the same manner.

The original equations (I) cannot be used because they contain an infinite number of unknown accelerations. By this transformation they have all been reduced to depend on three unknown quantities, viz. ω_x, ω_y, ω_z. But the equations thus obtained are so complicated as to be practically useless. To simplify them still further take axes *OA*, *OB*, *OC*, fixed in the body, and coincident with the principal axes of the body at the point *O*, and let ω_1, ω_2, ω_3 be the angular velocities about these axes.

Since the axes *Ox*, *Oy*, *Oz* are perfectly arbitrary, take them so that the axes *OA*, *OB*, *OC* are passing through them at the moment under consideration. Then $\omega_x=\omega_1$, $\omega_y=\omega_2$, $\omega_z=\omega_3$, and the last equation reduces to

$$\Sigma m\,(x^2+y^2)\,\frac{d\omega_z}{dt}+\Sigma m\,(x^2-y^2)\,\omega_1\omega_2=N.$$

We have now to find the relation between $\frac{d\omega_z}{dt}$ and $\frac{d\omega_3}{dt}$ *. Let *A*, *B*, *C* be the points in which the principal axes cut a sphere whose centre is at the fixed point. Let *OL* be any other axis, and let Ω be the angular velocity about it. Let the angles *LOA*, *LOB*, *LOC* be called respectively α, β, γ. Then by Art. 101

$$\Omega=\omega_1\cos\alpha+\omega_2\cos\beta+\omega_3\cos\gamma\,;$$

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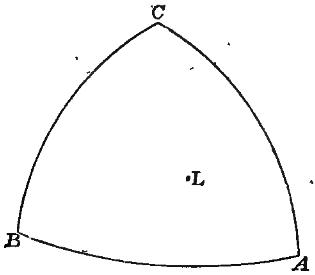
* This demonstration of the equality of $\frac{d\omega_z}{dt}$ and $\frac{d\omega_3}{dt}$ is due to the late Professor Slesser, of Queen's College, Belfast.

$$\therefore \frac{d\Omega}{dt} = \frac{d\omega_1}{dt}\cos\alpha + \frac{d\omega_2}{dt}\cos\beta + \frac{d\omega_3}{dt}\cos\gamma$$

$$- \omega_1 \sin\alpha \frac{d\alpha}{dt} - \omega_2 \sin\beta \frac{d\beta}{dt} - \omega_3 \sin\gamma \frac{d\gamma}{dt}.$$

Now let the line OL be fixed in space and coincide with OC



at the moment under consideration. Then $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$, $\gamma = 0$; therefore

$$\frac{d\Omega}{dt} = \frac{d\omega_3}{dt} - \omega_1 \frac{d\alpha}{dt} - \omega_2 \frac{d\beta}{dt}.$$

Also $\frac{d\alpha}{dt}$ is the rate at which A separates from a *fixed* point at C, which is clearly ω_2. Similarly $\frac{d\beta}{dt} = -\omega_1$. Hence

$$\frac{d\Omega}{dt} = \frac{d\omega_3}{dt}.$$

Thus $$\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt}, \quad \frac{d\omega_y}{dt} = \frac{d\omega_2}{dt}, \quad \frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}.$$

The equation $\frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}$ may appear at first sight to be a mere truism, but it is not so; ω_3 denotes the angular velocity of the body about the axis OC *fixed in the body*, ω_z denotes the angular velocity about a line Oz *fixed in space*, and determined by the condition that *at the time t*, OC coincides with it. At the time $t + \delta t$, OC will have separated from Oz, and we cannot therefore assert *a priori* that the angular velocity about OC will continue to be the same as that about Oz. The above investigation proves that this is the case as far as the first order of small quantities.

Substituting for $\frac{d\omega_z}{dt}$ the equation of motion becomes

$$\Sigma m\,(x^2 + y^2)\,\frac{d\omega_3}{dt} + \Sigma m\,(x^2 - y^2)\,\omega_1\omega_2 = N.$$

Let A, B, C be the moments of inertia of the body about the principal axes at O; then the three equations of motion are

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 &= L \\ B\frac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1 &= M \\ C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 &= N \end{aligned}\right\} \ldots\ldots\ldots \text{(III)}.$$

These are called Euler's Equations.

108. *To determine the pressure on the fixed point.*

If $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity, the equations (II) reduce to the form

$$M\frac{d^2\bar{x}}{dt^2} = P + \Sigma mX,$$

and two similar equations. It is necessary to express $\frac{d^2\bar{x}}{dt^2}$ the acceleration along a fixed straight line in terms of ω_x, ω_y, ω_z. This has been already done, and we have

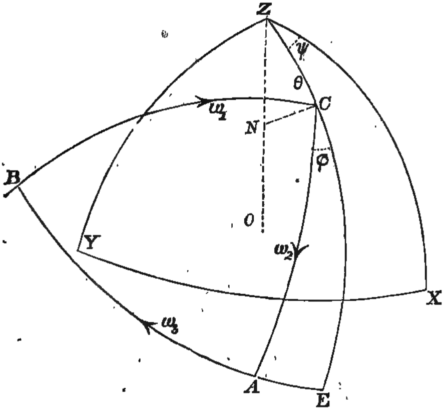
$$M\left\{\bar{z}\frac{d\omega_y}{dt} - \bar{y}\frac{d\omega_z}{dt} + \omega_y(\omega_x\bar{y} - \omega_y\bar{x}) - \omega_z(\omega_z\bar{x} - \omega_x\bar{z})\right\} = P + \Sigma mX,$$

and two similar equations.

109. *To determine the geometrical equations connecting the motion of the body in space with the angular velocities of the body about the three moving axes*, OA, OB, OC.

Let the fixed point O be taken as the centre of a sphere of radius unity; let X, Y, Z, A, B, C be the points in which the sphere is cut by the fixed and moving axes respectively. Let ZC, BA produced if necessary, meet in E. Let the angle $XZC = \psi$, $ZC = \theta$, $ECA = \phi$. It is required to determine the geometrical relations between θ, ϕ, ψ, and ω_1, ω_2, ω_3.

Draw CN perpendicular to OZ. Then since ψ is the angle the plane COZ makes with a plane XOZ fixed in space, the velocity of C perpendicular to the plane ZOC is $CN\frac{d\psi}{dt}$, which is the same as $\sin\theta\frac{d\psi}{dt}$, the radius OC of the sphere being unity. Also the velocity of C along ZC is $\frac{d\theta}{dt}$.



Thus the motion of C is represented by $\frac{d\theta}{dt}$ and $\sin\theta\frac{d\psi}{dt}$ respectively along and perpendicular to ZC. But the motion of C is also expressed by the angular velocities ω_1 and ω_2 respectively along BC and CA. These two representations of the same motion must therefore be equivalent. Hence resolving along and perpendicular to ZC we have

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \omega_1 \sin\phi + \omega_2 \cos\phi \\ \sin\theta \frac{d\psi}{dt} &= -\omega_1 \cos\phi + \omega_2 \sin\phi \end{aligned}\right\} \text{............ (IV).}$$

Similarly by resolving along CB and CA we have

$$\left.\begin{aligned} \omega_1 &= \frac{d\theta}{dt} \sin\phi - \frac{d\psi}{dt} \sin\theta \cos\phi \\ \omega_2 &= \frac{d\theta}{dt} \cos\phi + \frac{d\psi}{dt} \sin\theta \sin\phi \end{aligned}\right\} \text{............ (IV').}$$

These equations are precisely equivalent to (IV) and may be deduced from them by an algebraic transformation.

In the same way by drawing a perpendicular from E on OZ we may show that the velocity of E perpendicular to ZE is $\frac{d\psi}{dt} \sin ZE$, which is the same as $\frac{d\psi}{dt} \cos\theta$. Also the velocity of A relative to E along EA is in the same way $\frac{d\phi}{dt} \sin CA$, which is the same as $\frac{d\phi}{dt}$. Hence the whole velocity of A in space along AB is represented by $\frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}$. But this motion is also expressed by ω_3. As before these two representations of the same motion must be equivalent. Hence we have

$$\omega_3 = \frac{d\psi}{dt} \cos\theta + \frac{d\phi}{dt}.$$

If in a similar manner we had expressed the motion of any other point of the body as B, both in terms of ω_1, ω_2, ω_3 and θ, ϕ, ψ, we should have obtained other equations. But as we cannot have more than three independent relations, we should only arrive at equations which are algebraic transformations of those already obtained.

On Moving Axes.

110. In many cases it will be found convenient to refer the motion of the body under consideration to axes moving in space in some manner about a fixed origin. If we refer the motion of these axes to other axes fixed in space we shall have the inconvenience of two sets of axes. For this reason their motion at any instant is sometimes defined by angular velocities ($\theta_1, \theta_2, \theta_3$) about themselves. We are to regard the axes as if they were a material system of three straight lines at right angles whose motion at any instant is given by three coexistent angular velocities about axes instantaneously coincident with them.

When the axes are moving we may suppose the motion of the body to be determined by the three angular velocities $\omega_1, \omega_2, \omega_3$ about the axes, in the same manner as if the axes were fixed for an instant in space. The position of the body at the time $t + dt$ may be constructed from that at the time t by turning the body through the angles $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$ successively round the instantaneous position of the axes. But it must be remembered that $\omega_3 dt$ does not now give the angle the body has been turned through relative to the plane xz, but relative to some plane fixed in space passing through the instantaneous position of the axis of z. The angle turned through relative to the plane of xz is $(\omega_3 - \theta_3)\, dt$.

111. *To find the resolved part of the velocity of any particle parallel to the moving axes.*

The resolved parts of the velocity of any point whose co-ordinates are (x, y, z) are not given by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. These are the resolved velocities of the particle *relatively* to the axes. To find the motion in space we must add to these the resolved velocities due to the motion of the axes themselves. If we supposed the particle to be rigidly connected with the axes, it is clear that its velocities would be expressed by the forms given in Art. 105 with $\theta_1, \theta_2, \theta_3$ substituted for ω_1, ω_2, ω_3. So that the actual resolved velocities of the particle are

$$u = \frac{dx}{dt} - y\theta_3 + z\theta_2,$$

$$v = \frac{dy}{dt} - z\theta_1 + x\theta_3,$$

$$w = \frac{dz}{dt} - x\theta_2 + y\theta_1.$$

112. *To find the accelerations of any particle parallel to the axes we may proceed thus.*

The velocities of the particle at the time t resolved parallel to the axes Ox, Oy, Oz are respectively (u, v, w). At the time $t + dt$, the axes have been turned into the position Ox', Oy', Oz' by rotations equal to $\theta_1\,dt$, $\theta_2\,dt$, $\theta_3\,dt$ round the axes Ox, Oy, Oz respectively, and the velocities of the particle parallel to the axes in their new position are

$$u + \frac{du}{dt}dt,\quad v + \frac{dv}{dt}dt,\quad w + \frac{dw}{dt}dt.$$

To find the acceleration parallel to the axis of z we must resolve all these in that direction. By the rotation round the axis of y, the axis of x' has receded from the axis of z by an angle $\theta_2\,dt$, and by the rotation round the axis of x the axis of y' has approached the axis of z by an angle $\theta_1\,dt$. Also the axis of z' makes with the axis of z an angle whose cosine may be taken as unity. Therefore the resolved part of the velocity of the particle at the time $t + dt$ along the axis of z is ultimately

$$w + \frac{dw}{dt}dt - u\theta_2\,dt + v\theta_1\,dt.$$

But the acceleration is by definition, the ratio of the velocity gained in any time dt to that time. Hence if z be the acceleration resolved parallel to the axis of z, we have

$$Z = \frac{dw}{dt} - u\theta_2 + v\theta_1.$$

Similarly if X and Y be the accelerations parallel to the axes of x and y, we have

$$X = \frac{dw}{dt} - v\theta_3 + w\theta_2,$$

$$Y = \frac{dv}{dt} - w\theta_1 + u\theta_3.$$

113. *To express the geometrical conditions that a straight line whose equations with reference to the moving axes are given is fixed in direction in space.*

Let the equation to the given straight line be

$$\frac{x-f}{p} = \frac{y-g}{q} = \frac{z-h}{r},$$

and let the equations be so prepared that (p, q, r) are the direction-cosines of the line. Let a straight line be drawn through the origin parallel to this given straight line and let a point P be taken on this at any given distance L from the origin O. Then the co-ordinates of L are pL, qL, rL respectively. Since the straight line OL is fixed in direction in space, the resolved parts of the velocity of P parallel to the axes are zero. Hence we have

$$\frac{dLp}{dt} - Lq\theta_3 + Lr\theta_2 = 0,$$

and two similar equations. The required geometrical conditions are therefore

$$\frac{dp}{dt} - q\theta_3 + r\theta_2 = 0,$$

$$\frac{dq}{dt} - r\theta_1 + p\theta_3 = 0,$$

$$\frac{dr}{dt} - p\theta_2 + q\theta_1 = 0.$$

If the line OL be taken successively to coincide with OZ, OX, OY, axes fixed in space, these equations will be found to lead to those of Art. 109.

When it is necessary to refer the motion of these moving axes to other axes fixed in space, we may either use the equations of this article or those of Art. 109. Taking the notation of the article referred to, it is obvious (the axes being treated as a body consisting simply of three straight lines) that we shall have the results

$$\left.\begin{aligned} \frac{d\psi}{dt}\sin\theta &= -\theta_1\cos\phi + \theta_2\sin\phi \\ \frac{d\theta}{dt} &= \theta_1\sin\phi + \theta_2\cos\phi \\ \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt} &= \theta_3 \end{aligned}\right\}.$$

These equations will determine θ, ϕ, ψ in terms of the angular velocities θ_1, θ_2, θ_3.

114. *To express the geometrical conditions that a point whose co-ordinates with reference to the moving axes are* (x, y, z) *is fixed in space.*

This may be done by equating to zero the resolved parts of the velocity of the point as given in Art. 111. If the origin of the moving axes be fixed, the conditions are

$$\frac{dx}{dt} - y\theta_3 + z\theta_2 = 0,$$

and two similar equations. If the origin be in motion, let u_0, v_0, w_0 be the resolved parts of its velocity parallel to the axes, then the required conditions are clearly

$$u_0 + \frac{dx}{dt} - y\theta_3 + z\theta_2 = 0,$$

and two similar equations.

115. Let the direction-cosines of a straight line OM fixed relatively to the moving axes be (λ, μ, ν) and let it be required to refer the motion of OM to some straight line OL fixed in space whose direction-cosines at the time t are (p, q, r). Let the angle LOM be θ and let ψ be the angle the plane

LOM makes with any fixed plane in space passing through OL. Then it may be shown that

$$\left.\begin{aligned}&\cos\theta = p\lambda + q\mu + r\nu,\\ &\sin^2\theta\frac{d\psi}{dt} = \theta_1(p-\lambda\cos\theta) + \theta_2(q-\mu\cos\theta) + \theta_3(r-\nu\cos\theta)\end{aligned}\right\}.$$

If θ_l, θ_m be the resolved parts of the angular velocities about OL, OM respectively, the last equation may be written in the form

$$\sin^2\theta\frac{d\psi}{dt} = \theta_l - \theta_m\cos\theta.$$

If the straight line OM be not fixed relatively to the axes, then (λ, μ, ν) will be variable and we must add to the right-hand side of the second equation

$$\left(\lambda\frac{d\mu}{dt} - \mu\frac{d\lambda}{dt}\right)r + \left(\mu\frac{d\nu}{dt} - \nu\frac{d\mu}{dt}\right)p + \left(\nu\frac{d\lambda}{dt} - \lambda\frac{d\nu}{dt}\right)q.$$

116. *To explain a method of changing from fixed to moving axes.*

If a body be moving about a fixed point and we have established any *general* proposition referring its motion to fixed axes meeting at the fixed point, then we may use the following method to infer the corresponding proposition referring the motion to axes moving in any proposed manner about the origin. Suppose the general equation established to be

$$\psi\left\{\omega_x, \frac{d\omega_x}{dt}, \text{\&c.}\ldots.\right\} = 0,$$

where ω_x, ω_y, ω_z are the angular velocities about the fixed axes. Let ω_1, ω_2, ω_3 be the angular velocities of the body about the moving axes and let the motion of the axes be defined by the angular velocities θ_1, θ_2, θ_3 about themselves.

The fixed axes being arbitrary in position, let them be so chosen that, at the moment under consideration, the three moving axes are passing through them, so that the two sets are for an instant coincident. Then, by referring to Art. 110, we see that we may write $\omega_x = \omega_1$, $\omega_y = \omega_2$, $\omega_z = \omega_3$, but we cannot assert $\frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}$, for the moving axes at the time $t + dt$ are not coincident with the fixed axes.

To determine the relation between $\frac{d\omega_z}{dt}$ and $\frac{d\omega_3}{dt}$ we may proceed thus. Let OL be any straight line *fixed in space* making with the moving axes the angles α, β, γ. Let Ω be the angular velocity of the body about this straight line. Then as in page 165

$$\Omega = \omega_1 \cos\alpha + \omega_2 \cos\beta + \omega_3 \cos\gamma,$$

$$\therefore \frac{d\Omega}{dt} = \frac{d\omega_1}{dt}\cos\alpha + \frac{d\omega_2}{dt}\cos\beta + \frac{d\omega_3}{dt}\cos\gamma$$

$$- \omega_1 \sin\alpha \frac{d\alpha}{dt} - \omega_2 \sin\beta \frac{d\beta}{dt} - \omega_3 \sin\gamma \frac{d\gamma}{dt}.$$

Since OL is any fixed line in space, let it be so chosen that the moving axis of z coincides with it at the time t. Then $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$, and $\gamma = 0$, also $\frac{d\Omega}{dt}$ will be $\frac{d\omega_z}{dt}$. Since α is the angle OL makes with the moving axis of x, $\frac{d\alpha}{dt}$ is the rate at which the axis of x is separating from a fixed straight line coincident with the axis of z and this is clearly θ_2. Similarly $\frac{d\beta}{dt} = -\theta_1$, hence $\frac{d\omega_z}{dt} = \frac{d\omega_3}{dt} - \omega_1\theta_2 + \omega_2\theta_1$.

Similarly

$$\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt} - \omega_2\theta_3 + \omega_3\theta_2,$$

$$\frac{d\omega_y}{dt} = \frac{d\omega_2}{dt} - \omega_3\theta_1 + \omega_1\theta_3.$$

If we substitute these expressions in the given general equation

$$\psi\left\{\omega_x, \frac{d\omega_x}{dt}, \dots\right\} = 0,$$

we shall have the corresponding equation referred to moving axes.

If the moving axes be fixed in the body, and move with it, we have $\theta_1 = \omega_1$, $\theta_2 = \omega_2$, $\theta_3 = \omega_3$. In this case the relations will become $\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt}$, $\frac{d\omega_y}{dt} = \frac{d\omega_2}{dt}$, $\frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}$, as in Art. 107.

The preceding proof of the relation between $\frac{d\omega_3}{dt}$ and $\frac{d\omega_z}{dt}$ is a simple corollary from the parallelogram of angular velocities. The result will therefore be true for any other magnitude which obeys the "parallelogram law." In fact the demonstration is exactly the same. Now linear velocities and linear accelerations do obey this law. Hence the expressions obtained in Arts. 111, 112, for the velocities (u, v, w) and the accelerations (X, Y, Z) may be deduced from the one proved above.

If the general equation $\psi = 0$ should contain the velocity or acceleration of any particle of the body, then to obtain the corresponding equation referred to moving axes, we must substitute for these velocities or accelerations the expressions found in Arts. 111 and 112.

117. If the general equation should contain $\frac{d^2\omega_z}{dt^2}$ or any other second differential coefficients, the expressions to be substituted for them become more complicated.

Since $\frac{d\omega_x}{dt}$, $\frac{d\omega_y}{dt}$, $\frac{d\omega_z}{dt}$, being angular accelerations, follow the parallelogram law, we have

$$\begin{aligned}\frac{d\Omega}{dt} = &\left(\frac{d\omega_1}{dt} - \omega_2\theta_3 + \omega_3\theta_2\right)\cos\alpha \\ &+ \left(\frac{d\omega_2}{dt} - \omega_3\theta_1 + \omega_1\theta_3\right)\cos\beta \\ &+ \left(\frac{d\omega_3}{dt} - \omega_1\theta_2 + \omega_2\theta_1\right)\cos\gamma.\end{aligned}$$

We may repeat the same reasoning and we shall finally obtain

$$\begin{aligned}\frac{d^2\omega_z}{dt^2} = \frac{d}{dt}\left\{\frac{d\omega_3}{dt} - \omega_1\theta_2 + \omega_2\theta_1\right\} \\ - \theta_2\left\{\frac{d\omega_1}{dt} - \omega_2\theta_3 + \omega_3\theta_2\right\} \\ + \theta_1\left\{\frac{d\omega_2}{dt} - \omega_3\theta_1 + \omega_1\theta_3\right\}.\end{aligned}$$

So we may proceed to treat third and higher differential coefficients.

118. *A body is turning about a fixed point in any manner, to determine the moments of the momentum about the axes,* i. e. *to find the areas conserved round those axes.* See Chap. II. Art. 44.

Let (x, y, z) be the co-ordinates of any particle m of the body referred to axes fixed in space meeting at the fixed point. Let $\omega_x, \omega_y, \omega_z$ be the angular velocities of the body about the fixed axes.

Then the moment of the momentum about the axis of z is

$$h_3 = \Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right).$$

Substituting for $\frac{dx}{dt}, \frac{dy}{dt}$ their values

$$\left.\begin{aligned}\frac{dx}{dt} &= \omega_y z - \omega_z y \\ \frac{dy}{dt} &= \omega_z x - \omega_x z\end{aligned}\right\},$$

we have

$$h_3 = \Sigma m\,(x^2 + y^2)\,\omega_z - (\Sigma mxz)\,\omega_x - (\Sigma myz)\,\omega_y.$$

The coefficients of $\omega_x, \omega_y, \omega_z$ are the moments and products of inertia of the body about the axes, and if the axes be fixed in space, these will generally be variable. In some cases it will be found more convenient to take as axes of reference three straight lines fixed in the body.

Let ω_1, ω_2, ω_3 be the angular velocities of the body about rectangular axes Ox', Oy', Oz' fixed in the body and meeting at the fixed point O. Since in the expression given above for h_3 the fixed axes may be any whatever, let them be chosen so that the moving axes coincide with them at the time t. Then, as in the proposition of Art. 116, the moment of the momentum about the moving axis of z will be expressed by the form

$$h_3' = C\omega_3 - E\omega_1 - D\omega_2,$$

where

$$C = \Sigma m\,(x'^2 + y'^2), \quad E = \Sigma m x'z', \quad D = \Sigma m y'z'.$$

These will be constant throughout the motion, and their values may be found by the rules given in Chapter I.

If the axes fixed in the body be principal axes, then the products of inertia will vanish. The expressions for the moments of the momentum will then take the simple forms

$$\left.\begin{aligned} h_1' &= A\omega_1 \\ h_2' &= B\omega_2 \\ h_3' &= C\omega_3 \end{aligned}\right\},$$

where A, B, C are the principal moments of the body.

Let the direction-cosines of the moving axes with reference to axes fixed in the body be given by the following diagram; where, for example, b_3 is the cosine of the angle between the axes of z and y'. It has just been proved that the resultant of the momenta of all the particles of the body is equivalent to the three "couples" h_1', h_2', h_3' about the axes Ox', Oy', Oz'. Hence the moment of the momentum about the axis of z which is fixed in space may be written in the form

	x,	y,	z
x'	a_1,	a_2,	a_3
y'	b_1,	b_2,	b_3
z'	c_1,	c_2,	c_3

$$h_3 = h_1'a_3 + h_2'b_3 + h_3'c_3,$$

which will be frequently found useful.

It may be required to find the moment of the momentum about axes neither fixed in space nor in the body, but moving in any arbitrary manner. By the proposition of Art. 116, we see that this will be expressed by the same form as if the axes were fixed. So that the moment required is

$$= \Sigma m (x^2 + y^2) \omega_3 - (\Sigma mxz) \omega_1 - (\Sigma myz) \omega_2 .$$

If the axis of z coincide with the instantaneous axis of rotation, $\omega_x = 0$, $\omega_y = 0$, and ω_z is the resultant angular velocity. The expressions for the moments of the momentum or areas conserved about the axes of x, y, z become respectively

$$- (\Sigma mxz) \omega_z$$

$$- (\Sigma myz) \omega_z ,$$

$$\Sigma m (x^2 + y^2) \omega_z .$$

The axis of the couple which is the resultant of the moments of the momentum about the axis is sometimes called the resultant axis of areas. It is to be remarked that this axis does not in general coincide with the instantaneous axis of rotation. The two are coincident only when the axis of rotation is a principal axis.

If it be required to find the moment of the momentum about the axis of z of a rigid body moving in any manner in space, we may use the principle proved in Chapter II. Art. 43.

In the case of a system of rigid bodies, the moment of their momenta may be found by adding up the separate moments of the several bodies.

119. *A body is turning about a fixed point in any manner, to determine the moments of the effective forces about the axes.*

Let (x, y, z) be the co-ordinates of any particle m of the body referred to axes fixed in space and meeting at the fixed point, and let h_1, h_2, h_3 be the moments of the momentum about the axes.

The moment of the effective forces about the axis of z is

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right),$$

and this may be written in the form $\frac{dh_3}{dt}$.

Thus the moments of the effective forces about axes Ox, Oy, Oz fixed in space are respectively

$$\frac{dh_1}{dt},\ \frac{dh_2}{dt},\ \frac{dh_3}{dt},$$

where h_1, h_2, h_3 have the values found by the last proposition.

Let h_1', h_2', h_3' be the moments of the momentum, found by the last Art., about axes Ox', Oy', Oz' moving in space about the fixed origin. Let θ_1, θ_2, θ_3 be the angular velocities of these axes about their instantaneous directions. Then since moments or couples follow the parallelogram law, we see by the proposition of Art. 116 that the moments of the effective forces about the moving axes are respectively

$$\frac{dh_1'}{dt}-h_2'\theta_3+h_3'\theta_2,$$

$$\frac{dh_2'}{dt}-h_3'\theta_1+h_1'\theta_3,$$

$$\frac{dh_3'}{dt}-h_1'\theta_2+h_2'\theta_1,$$

where h_1', h_2', h_3' have the values given by the last proposition.

If the moving axes be *fixed in the body*, we have $\theta_1=\omega_1$, $\theta_2=\omega_2$, $\theta_3=\omega_3$, and the equations admit of some simplification. If the axes be the *principal axes* we have $h_1'=A\omega_1$, $h_2'=B\omega_2$, $h_3'=C\omega_3$, and the moments of the effective forces take the simple forms

$$A\frac{d\omega_1}{dt}-(B-C)\,\omega_2\omega_3,$$

$$B\frac{d\omega_2}{dt}-(C-A)\,\omega_3\omega_1,$$

$$C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2,$$

where A, B, C are the principal moments. See Art. 107.

If it be required to find the moment about the axis of z of the effective forces on a rigid body moving in any manner in space, we may use the principle proved in Chap. II. Art. 43.

In the case of a system of rigid bodies, the moment of their effective forces may be found by adding up the separate moments of the several bodies.

120. *To obtain the general equations of motion of a system of rigid bodies.*

These equations have been already obtained in Chap. II. Art. 45, when the system is referred to axes fixed in space. If the axes be moveable we must replace the accelerations $\frac{du}{dt}$, $\frac{dv}{dt}$, $\frac{dw}{dt}$ by the corresponding forms in Art. 112 and the couples $\frac{dh_1}{dt}$, $\frac{dh_2}{dt}$, $\frac{dh_3}{dt}$ by the expressions in Art. 119.

Thus, suppose we refer the motion to three axes moving in space about a fixed origin O. Let X, Y, Z be the impressed forces on any rigid body of the system, including the unknown reactions of the other bodies of the system. Let L, M, N be the moments of these forces about axes drawn through the centre of gravity of the body parallel to the coordinate axes. Let m be the mass of the body. Then if we adopt the notation of Arts. 112 and 119, the equations of motion for the rigid body under consideration will be

$$\left.\begin{aligned}\frac{du}{dt}-v\theta_3+w\theta_2&=\frac{X}{m},\\ \frac{dv}{dt}-w\theta_1+u\theta_3&=\frac{Y}{m},\\ \frac{dw}{dt}-u\theta_2+v\theta_1&=\frac{Z}{m},\end{aligned}\right\}$$

and

$$\left.\begin{aligned}\frac{dh_1'}{dt} - h_2'\theta_3 + h_3'\theta_2 &= L,\\ \frac{dh_2'}{dt} - h_3'\theta_1 + h_1'\theta_3 &= M,\\ \frac{dh_3'}{dt} - h_1'\theta_2 + h_2'\theta_1 &= N,\end{aligned}\right\}$$

where h_1', h_2', h_3' have the values given in Art. 118.

Similar equations will apply for each body of the system.

Besides these dynamical equations there will be the geometrical equations expressing the connections of the system. As every such forced connection is accompanied by some reaction, the number of geometrical equations will be the same as the number of unknown reactions in the system.

Thus we have sufficient equations to determine the motion.

121. If two of the principal moments at the fixed origin O are equal, it is often convenient to choose as axis of z' the axis OC of unequal moment, and as axes of x', y' two other axes OA, OB moving in any manner round OC. Let χ be the angle the plane of AOC makes with some plane fixed in the body and passing through OC. Then we have $\theta_1 = \omega_1$, $\theta_2 = \omega_2$, and $\theta_3 = \omega_3 + \frac{d\chi}{dt}$. Also, by Art. 118, we have $h_1' = A\omega_1$, $h_2' = A\omega_2$, $h_3' = C\omega_3$. The equations of motion of Art. 120 now become

$$\left.\begin{aligned}A\left(\frac{d\omega_1}{dt} - \omega_2\frac{d\chi}{dt}\right) - (A-C)\,\omega_2\omega_3 &= L,\\ A\left(\frac{d\omega_2}{dt} + \omega_1\frac{d\chi}{dt}\right) + (A-C)\,\omega_3\omega_1 &= M,\\ C\frac{d\omega_3}{dt} \qquad\qquad\qquad\qquad &= N.\end{aligned}\right\}$$

In this case the most convenient geometrical equations to express the relations of these moving axes to straight lines fixed in space will be those given in Art. 109.

Since $\frac{d\chi}{dt}$ is arbitrary, it may be chosen to simplify either the dynamical or the geometrical equations.

First, we may put $\frac{d\chi}{dt} = -\omega_3$. The dynamical equations then become

$$A\frac{d\omega_1}{dt} + C\omega_2\omega_3 = L,$$

$$A\frac{d\omega_2}{dt} - C\omega_1\omega_3 = M,$$

$$C\frac{d\omega_3}{dt} \qquad\qquad = N.$$

Secondly, we may so choose $\frac{d\chi}{dt}$ that $\phi = 0$. In this case the plane COA always passes through a straight line OZ fixed in space. The geometrical equations then become

$$\frac{d\theta}{dt} = \omega_2, \qquad \frac{d\psi}{dt}\sin\theta = -\omega_1,$$

$$-\frac{d\chi}{dt} + \frac{d\psi}{dt}\cos\theta = \omega_3.$$

122. If three principal moments at the fixed origin O be equal, there are three sets of axes such that when the motion is referred to them, the equations take a simple form.

First. We may choose axes fixed in space. Since every axis is a principal axis in the body, the general equations of motion become

$$\frac{d\omega_1}{dt} = \frac{L}{A}, \quad \frac{d\omega_2}{dt} = \frac{M}{A}, \quad \frac{d\omega_3}{dt} = \frac{N}{A}.$$

The geometrical equations of Art. 109 are not required.

Secondly. We may choose one axis as that of OC fixed in space and let the other two move round it in any manner, then as in Art. 121, the equations of motion become

$$\left.\begin{aligned}\frac{d\omega_1}{dt}-\omega_2\frac{d\chi}{dt}&=\frac{L}{A}\\ \frac{d\omega_2}{dt}+\omega_1\frac{d\chi}{dt}&=\frac{M}{A}\\ \frac{d\omega_3}{dt}\qquad\qquad&=\frac{N}{A}\end{aligned}\right\}.$$

Thirdly. We can take as axes any three straight lines at right angles moving in space in any proposed manner. The equations of motion are then by Art. 120

$$\frac{d\omega_1}{dt}-\omega_2\theta_3+\omega_3\theta_2=\frac{L}{A},$$

$$\frac{d\omega_2}{dt}-\omega_3\theta_1+\omega_1\theta_3=\frac{M}{A},$$

$$\frac{d\omega_3}{dt}-\omega_1\theta_2+\omega_2\theta_1=\frac{N}{A}.$$

The geometrical equations will then be the same as those given in Art. 109 or Art. 113.

On Motion relative to the Earth.

123. The motion of a body on the surface of the earth is not exactly the same as if the earth were at rest. As an illustration of the use of the equations of this chapter, we shall proceed to determine the equations of motion of a particle referred to axes of co-ordinates fixed in the earth and moving with it.

Let O be any point on the surface of the earth whose latitude is λ. Thus λ is the angle the normal to the surface of still water at O makes with the plane of the equator. Let the axis of z be vertical at O and measured positively in the direction opposite to gravity. Let the axes of x and y be respectively a tangent to the meridian and a perpendicular to it, their positive directions being respectively south and west. Let ω be the angular velocity of the earth, b the distance of the point O from the axis of rotation.

First, we may reduce the point O to rest by applying to the particle an acceleration $\omega^2 b$ tending from the axis of rotation. Now gravity at the point O is the resultant of the attraction of the earth and the centrifugal force $\omega^2 b$. In our case the force acting on the particle will be the attraction of the earth simply. Let X_0, Y_0, Z_0 be the resolved parts of the attraction of the earth at the point O; $X_0 + X$, $Y_0 + Y$, $Z_0 + Z$ the attractions on the particle. Then the resultant of X_0, Y_0, Z_0 and $\omega^2 b$ is gravity at O acting vertically. Let this be represented by g.

The motion of the axes is given by $\theta_1 = \omega \cos \lambda$, $\theta_2 = 0$, $\theta_3 = -\omega \sin \lambda$, since the earth turns from west to east. Hence by Art. 111 the actual velocities of any particle whose co-ordinates are (x, y, z) are

$$\left.\begin{aligned} u &= \frac{dx}{dt} + \omega \sin \lambda\, y \\ v &= \frac{dy}{dt} - \omega \cos \lambda\, z - \omega \sin \lambda\, x \\ w &= \frac{dz}{dt} + \omega \cos \lambda\, y \end{aligned}\right\},$$

and substituting these in the equations of Art. 120 the equations of motion are

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} + 2\omega \sin \lambda \frac{dy}{dt} - \omega^2 \sin^2 \lambda\, x - \omega^2 \sin \lambda \cos \lambda\, z = X \\ &\frac{d^2y}{dt^2} - 2\omega \cos \lambda \frac{dz}{dt} - 2\omega \sin \lambda \frac{dx}{dt} - \omega^2 y = Y \\ &\frac{d^2z}{dt^2} + 2\omega \cos \lambda \frac{dy}{dt} - \omega^2 \cos^2 \lambda\, z - \omega^2 \sin \lambda \cos \lambda\, x = -g + Z \end{aligned}\right\}.$$

These equations may be simplified if we neglect such small quantities as the difference between the force of gravity at different heights. If a be the equatoreal radius of the earth and g' the force of gravity at a height z, we have

$g' = g\left(1 - \frac{2z}{a}\right)$ nearly. Now $\omega^2 a$ is the centrifugal force at the equator, which is known to be $\frac{1}{289} g$. Hence if we neglect the small term $g\frac{z}{a}$ we must also neglect $\omega^2 z$. The equations will therefore become

$$\left.\begin{aligned}&\frac{d^2x}{dt^2} + 2\omega \sin\lambda \frac{dy}{dt} = X\\ &\frac{d^2y}{dt^2} - 2\omega\cos\lambda\frac{dz}{dt} - 2\omega\sin\lambda\frac{dx}{dt} = Y\\ &\frac{d^2z}{dt^2} + 2\omega\cos\lambda\frac{dy}{dt} = -g + Z\end{aligned}\right\},$$

where the terms (X, Y, Z) include all the accelerating forces, except gravity, which act on the particle. These equations agree with those given by Poisson, *Journal Polytechnique* 1838.

As an example, let us consider the case of a particle dropped from a height h. The initial conditions are therefore $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ all zero and $z = h$. As a first approximation, neglect all the terms containing the small factor ω. Then we have $x = 0$, $y = 0$, $z = h - \frac{1}{2}gt^2$.

For a second approximation, we may substitute these values of (x, y, z) in the small terms. We have after integration

$$x = 0,\ y = -\omega\cos\lambda g\frac{t^3}{3},\ z = h - \frac{1}{2}gt^2.$$

Thus there will be a small deviation towards the east, proportional to the cube of the time of descent. There will be no southerly deviation, and the vertical motion will be the same as if the earth were at rest.

124. For the determination of the motion of a projectile it will be found convenient to refer the motion to axes more

generally placed. Let the axis of z be vertical as before and let the axis of x make an angle β with the meridian, the angle being measured from the south towards the west. Then $\theta_1 = \omega \cos\lambda \cos\beta$, $\theta_2 = -\omega\cos\lambda\sin\beta$, $\theta_3 = -\omega\sin\lambda$. The resolved parts of the velocity of the particle are

$$\left.\begin{aligned} u &= \frac{dx}{dt} - y\theta_3 + z\theta_2, \\ v &= \frac{dy}{dt} - z\theta_1 + x\theta_3, \\ w &= \frac{dz}{dt} - x\theta_2 + y\theta_1, \end{aligned}\right\}$$

and the equations of motion become

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} - 2\frac{dy}{dt}\theta_3 + 2\frac{dz}{dt}\theta_2 = X \\ &\frac{d^2y}{dt^2} - 2\frac{dz}{dt}\theta_1 + 2\frac{dx}{dt}\theta_3 = Y \\ &\frac{d^2z}{dt^2} - 2\frac{dx}{dt}\theta_2 + 2\frac{dy}{dt}\theta_1 = -g + Z \end{aligned}\right\}.$$

These equations may be solved in any particular case by the method of continued approximation. If we neglect the small terms we get a first approximation to the values of (x, y, z). To find a second approximation we may substitute these values in the terms containing ω and integrate the resulting equations. As these equations are only true on the supposition that ω^2 may be neglected, we cannot proceed to a third approximation.

125. In many cases it will be found advantageous to refer the motion to axes not fixed in the earth but moving in some known manner. Let the axis of z be vertical as before and let the axes of x and y move slowly round the vertical with angular velocity $\omega \sin\lambda$ in the direction from the south towards the west. In this case we have

$$\theta_1 = \omega\cos\lambda\cos\beta,\ \ \theta_2 = -\omega\cos\lambda\sin\beta,$$

and $$\theta_3 = -\omega\sin\lambda + \omega\sin\lambda = 0,$$

where β is the angle the axis of x makes with the tangent to the meridian, so that $\frac{d\beta}{dt} = \omega \sin \lambda$. If, as before, we neglect quantities which contain the square of ω as a factor, the terms which contain $\frac{d\theta_1}{dt}$ and $\frac{d\theta_2}{dt}$ must be omitted. Hence the required equations may be obtained from those of Art. 124, by putting $\theta_3 = 0$. We thus obtain

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2\omega \cos \lambda \sin \beta \frac{dz}{dt} &= X \\ \frac{d^2y}{dt^2} - 2\omega \cos \lambda \cos \beta \frac{dz}{dt} &= Y \\ \frac{d^2z}{dt^2} + 2\omega \cos \lambda \sin \beta \frac{dx}{dt} + 2\omega \cos \lambda \cos \beta \frac{dy}{dt} &= -g + Z \end{aligned}\right\} .$$

If the particle be constrained to move in a horizontal plane the reaction of the plane will enter only into the last equation and the motion will be determined by the first two equations. In this case since z is constant, the equations will be independent of ω. It follows that the motion of such a particle relative to these moving axes will be exactly the same as if the earth had been at rest and the motion referred to fixed axes, the *relative* initial conditions being the same in both cases. If then we calculate the path without regard to the rotation of the earth, taking the initial conditions relative to the moving axes, the effect of the rotation of the earth may be allowed for by making this path turn round the vertical with uniform angular velocity $\omega \sin \lambda$*.

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* The steps of the reasoning by which this result has been arrived at may be stated thus:

(1) A point O in the horizontal plane is taken to be on the surface of the earth and is reduced to rest. It is then proved that the particle while in the neighbourhood of O is acted on by its weight in a direction normal to the surface of the earth.

(2) The earth is now turning round an axis through O parallel to the axis of figure with angular velocity ω. This angular velocity is resolved into two, viz. $\theta_1 = \omega \cos \lambda$ round a tangent to the meridian, and $\theta_3 = -\omega \sin \lambda$ round the vertical at O. Now θ_1 and θ_2 are so small that their product may be neglected. Hence by the principle of the superposition of small motions we may determine their whole effect by adding together the effects produced by each separately.

126. Let us apply these equations to determine the motion of a simple pendulum of mass m and length l. Let mT be the tension of the string. The equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2\omega\cos\lambda\sin\beta\frac{dz}{dt} &= -T\frac{x}{l} \\ \frac{d^2y}{dt^2} - 2\omega\cos\lambda\cos\beta\frac{dz}{dt} &= -T\frac{y}{l} \\ \frac{d^2z}{dt^2} + 2\omega\cos\lambda\sin\beta\frac{dx}{dt} + 2\omega\cos\lambda\cos\beta\frac{dy}{dt} &= -g - T\frac{z}{l} \end{aligned}\right\},$$

the origin being taken at the point of suspension.

If the oscillation be sufficiently small, z will differ from l by small quantities of the order α^2 where α is the semi-angle of oscillation. If then we may reject terms of the order $\alpha^2\omega$ in the expressions for the accelerations, the equations become after elimination of T

$$\left.\begin{aligned} x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} &= 0 \\ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 &= C - 2gz \\ z &= -\sqrt{l^2 - x^2 - y^2} \end{aligned}\right\},$$

from which all trace of ω has disappeared.

The small oscillations of a pendulum on the earth referred to axes turning round the vertical with angular velocity

(3) It is a known theorem that if a particle be constrained to move in a plane which turns round any axis in that plane with an angular velocity θ_1, the motion in that plane may be found by regarding the plane as fixed and impressing an angular velocity $\theta_1{}^2y$ on the particle, where y is the distance of the particle from the axis. This impressed force is neglected because it depends on the square of ω. Hence the angular velocity θ_1 has no sensible effect.

(4) The angular velocity $\theta_3 = -\omega\sin\lambda$ round the normal to the plane, does not in any manner affect the real motion in space of the particle. If we refer the motion to axes which move relatively to the plane with an angular velocity round $O = \omega\sin\lambda$, such axes would be fixed in space and the motion, with the same relative initial conditions, will be the same as if the earth were at rest.

The small oscillations of a pendulum in any latitude may be deduced from this theorem.

$\omega \sin \lambda$ are therefore the same as those of an imaginary pendulum suspended from an absolutely fixed point.

Let us then suppose the pendulum to be drawn aside from the vertical at any small angle α and then let go. Relative therefore to the axes moving round the vertical with angular velocity $\omega \sin \lambda$ we must suppose the particle to be projected with a velocity $l \sin \alpha\omega \sin \lambda$ perpendicular to the initial plane of displacement. It is then known* to describe an ellipse very nearly, the ratio of the axes being $\omega \sin \lambda \sqrt{\frac{l}{g}}$. We see that the effect of the rotation of the earth is to make this ellipse turn round the vertical with uniform angular velocity $\omega \sin \lambda$ in a direction from south to west. If the angle α be not so small that its square may be neglected, it is known by Dynamics of a particle that independent of all considerations of the rotation of the earth there will be a progression of the apsides of the ellipse. It is therefore necessary for the success of the experiment that the length l of the pendulum should be very long. This motion of the apsides depending on the magnitude of α is in the opposite direction to that caused by the rotation of the earth and can therefore be always distinguished from it.

It also appears that the time of oscillation is unaffected by the rotation of the earth, provided the arc of oscillation be so small that the effects of forces whose magnitude contains the factor $\omega\alpha^2$ may be neglected.

EXAMPLES.

1. The locus of points in a body moving about a fixed point which at any proposed instant have the same actual velocity is a circular cylinder. See Art. 97.

2. The geometrical motion of a body is represented by angular velocities inversely proportional to $\beta - \gamma$, $\gamma - \alpha$, $\alpha - \beta$ about three lines forming three edges of a cube which do not

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* This may be easily deduced from the equations of Art. 126 by putting $z = -l$.

meet nor are parallel. Prove that the body rotates about the line

$$(\beta - \gamma)\, x - a\alpha = (\gamma - \alpha)\, y - a\beta = (\alpha - \beta)\, z - a\gamma,$$

$2a$ being an edge of the cube, the centre being the origin, and the axes parallel to the edges.

3. A body has an angular velocity ω about the axis

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

where $l^2 + m^2 + n^2 = 1$. The motion is equivalent to rotations $l\omega$, $m\omega$, $n\omega$ about the co-ordinate axes, and translations $(m\gamma - n\beta)\,\omega$, $(n\alpha - l\gamma)\,\omega$, $(l\beta - m\alpha)\,\omega$ in the directions of the axes.

This follows from the analogy of forces in Statics to angular velocities in Dynamics. See Art. 104.

4. A body has equal angular velocities about two axes which neither meet nor are parallel. Prove that the central axis of the motion is equally inclined to each of the axes.

5. A body is referred to rectangular axes x, y, z, and the origin remaining the same the axes are changed to x', y', z', according to the scheme in the margin. Show that this is equivalent to turning the body round an axis whose equations are any two of the following three:

	x',	y',	z'
x	a_1,	a_2,	a_3
y	b_1,	b_2,	b_3
z	c_1,	c_2,	c_3

$$(a_1 - 1)\, x + a_2 y + a_3 z = 0,$$

$$b_1 x + (b_2 - 1)\, y + b_3 z = 0,$$

$$c_1 x + c_2 y + (c_3 - 1)\, z = 0,$$

through an angle θ, where

$$3 - 4 \sin^2 \frac{\theta}{2} = a_1 + b_2 + c_3.$$

What is the condition that these three equations are consistent?

Take two points one on each of the axes of z and z' at a distance h from the origin. Their co-ordinates are $(0, o, h)$ $(a_3 h, b_3 h, c_3 h)$ therefore their distance is

$\sqrt{2(1-c_3)}\,h$. But it is also $2h\sin\gamma\sin\frac{\theta}{2}$; $\therefore\ 2\sin^2\frac{\theta}{2}\sin^2\gamma=1-c_3$. We have by similar reasoning $2\sin^2\frac{\theta}{2}\sin^2\alpha=1-a_1$ and $2\sin^2\frac{\theta}{2}\sin^2\beta=1-b_2$, whence the equation to find θ follows at once.

6. If in a rigid body moving in any manner about a fixed point a series of points be taken along any straight line in the body, and through these points straight lines be drawn in the directions of the instantaneous motion of the points, prove that the locus of these straight lines is an hyperbolic paraboloid.

7. A body is moved from any position in space to any other, and every point of the body in the first position is joined to the same point in the second position. If all the straight lines thus found be taken which pass through a given point, they will form a cone of the second order. Also if the middle points of all these lines be taken, they will together form a body capable of an infinitesimal motion, each point of it along the line on which the same is situate. Cayley's *Report to the Brit. Assoc.*, 1862.

α. A particle is projected with velocity V in a direction making an angle α with the horizontal plane, and such that the vertical plane through the direction of projection makes an angle β with the plane of the meridian, the angle β being measured from the south towards the west. If x be measured horizontally in the plane of projection, y be measured horizontally in a direction making an angle $\beta+\frac{\pi}{2}$ with the meridian, and z vertically upwards from the point of projection, prove that

$$x=V\cos\alpha t+\left(V\sin\alpha t^2-\frac{1}{3}gt^3\right)\omega\cos\lambda\sin\beta,$$

$$y=\left(V\sin\alpha t^2-\frac{1}{3}gt^3\right)\omega\cos\lambda\cos\beta+V\cos\alpha t^2\omega\sin\lambda,$$

$$z=V\sin\alpha t-\frac{1}{2}gt^2-V\cos\alpha t^2\omega\cos\lambda\sin\beta,$$

where λ is the latitude of the place, and ω the angular velocity of the earth about its axis of figure.

Show also that the increase of range on the horizontal plane through the point of projection is

$$4\omega \frac{V^3}{g^2} \sin\beta \cos\lambda \sin\alpha \left(\frac{1}{3} \sin^2\alpha - \cos^2\alpha\right),$$

and the deviation to the right of the plane of projection is

$$4\omega \frac{V^3}{g^2} \sin^2\alpha \left(\cos\lambda \cos\beta \frac{\sin\alpha}{3} + \sin\lambda \cos\alpha\right).$$

9. A bullet is projected from a gun nearly horizontally with great velocity so that the trajectory is nearly flat, prove that the deviation is nearly equal to $Rt\omega \sin\lambda$, where R is the range, and the other letters have the same meaning as in the last question. The deviation is always to the right of the plane of firing in the Northern hemisphere, and to the left in the Southern hemisphere. It is asserted (*Comptes Rendus*, 1866) that the deviation due to the earth's rotation as calculated by this formula is as much as half the actual deviation in Whitworth's gun.

10. A spherical bullet is projected with so great a velocity that the resistance of the air must be taken into account. The resistance of the air being assumed to be $\frac{(\text{vel.})^2}{k}$, and the trajectory to be flat, prove that

$$x = k \log\left(1 + \frac{Vt}{k}\right)$$

$$y = \frac{2\omega \sin\lambda}{V} k^2 \left(e^{\frac{x}{k}} - \frac{x}{k} - 1\right),$$

$$z = x \tan\alpha - \frac{gk^2}{4V^2}\left(e^{2\frac{x}{k}} - 2\frac{x}{k} - 1\right)$$

$$- \frac{2\omega \sin\beta \cos\lambda}{V} k^2 \left(e^{\frac{x}{k}} - \frac{x}{k} - 1\right).$$

These are given by Poisson, *Journal Polytechnique*, 1838.

11. A heavy particle is suspended from a *fixed* point of support by a string of length a. It performs elliptic oscillations whose major and minor semi-axes are b and c. If b and c be small compared with a, prove that the apses will advance through an angle $\frac{3}{8}\frac{bc}{a^2}2\pi$ nearly in each complete revolution of the particle. If b and c be not small compared with a but be very nearly equal, the apse will advance through an an angle

$$\left(\frac{1}{\sqrt{1-\frac{3}{4}\sin^2\alpha}}-1\right)2\pi,$$

where $\sin\alpha=\frac{b}{a}$ in each complete revolution of the particle.

12. A pendulum at rest relatively to the earth is projected in any direction with a small angular velocity, show that the oscillations will take place in a vertical plane turning uniformly round the vertical so that the pendulum becomes vertical once in each half oscillation.

13. Let θ be the angle a pendulum of length l makes with the vertical, and ϕ the angle the vertical plane containing the pendulum makes with a vertical plane which turns round the vertical with uniform angular velocity $\omega\sin\lambda$ in a direction from south to west. Prove that when terms depending on ω^2 are neglected the equations of motion become

$$\left(\frac{d\theta}{dt}\right)^2+\sin^2\theta\left(\frac{d\phi}{dt}\right)^2=\frac{2g}{l}\cos\theta+A,$$

$$\frac{d}{dt}\left(\sin^2\theta\frac{d\phi}{dt}\right)=2\sin^2\theta\cos(\phi+\beta)\,\omega\cos\lambda\frac{d\theta}{dt},$$

where A is an arbitrary constant, and the other letters have the meanings given to them in Art. 124. See M. Quet in Liouville's *Journal*, 1853.

These equations will be found convenient in treating the motion of a pendulum. They may be easily obtained by transforming those given in Art. 126 to polar co-ordinates.

CHAPTER VI.

ON MOMENTUM.

127. THE term Momentum has been given as the heading of this Chapter, though it only expresses a portion of its contents. The object of the Chapter may be enunciated in the following problem. The circumstances of motion of a system at any time t_0 are given. At the time t_1 the system is moving under other circumstances. It is required to determine the relations which may exist between these two motions. The manner in which these changes are effected by the forces is not the subject of enquiry. We only wish to determine what changes have been effected in the time $t_1 - t_0$. If the time $t_1 - t_0$ be very small, and the forces very great, this becomes the general problem of impulses. This also will be considered in the Chapter.

128. Let us refer the system to any fixed axes Ox, Oy, Oz. Then the six general equations of motion may by Art. 39 be written in the form

$$\left.\begin{aligned} \Sigma m \frac{d^2z}{dt^2} &= \Sigma m Z \\ \Sigma m \left(x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2}\right) &= \Sigma m (xY - yX) \end{aligned}\right\}.$$

Integrating these from $t = t_0$ to $t = t_1$, we have

$$\left[\Sigma m \frac{dz}{dt}\right]_{t_0}^{t_1} = \Sigma m \int_{t_0}^{t_1} Z\, dt,$$

$$\left[\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt}\right)\right]_{t_0}^{t_1} = \Sigma m \int_{t_0}^{t_1} (xY - yX)\, dt.$$

Let a force P act on a moving particle m during any time $t_1 - t_0$, and let this time be divided into intervals each equal to dt. At the middle of each of these intervals, let a line be drawn from the position of m at that instant, to represent the value of $mPdt$ at that instant, in direction and magnitude. Then the resultant of these forces, found by the rules of Statics, may be called the "whole force" expended in the time $t_1 - t_0$. Thus $\int_{t_0}^{t_1} mZdt$ is the whole force resolved parallel to the axis of Z. These equations then show that

(1) The change produced by any forces in the resolved part of the momentum of any system is equal in any time to the whole resolved force in that direction.

(2) The change produced by any forces in the moment of the momentum of the system about any straight line is equal in any time to the whole moment of these forces about that straight line.

When the interval $t_1 - t_0$ is very small, the "whole force" expended is the usual measure of an impulsive force, and the preceding equations are identical with those given in Art. 49 of Chapter II.

It is not necessary to deduce these two results from the equations of motion. The following general theorem, which is really equivalent to the two theorems enunciated above, may be easily obtained by an application of D'Alembert's principle.

129. If the momentum of any particle of a system in motion be compounded and resolved, as if it were a force acting at the instantaneous position of the particle, according to the rules of Statics, then the momenta of all the particles at any time t_1 are together equivalent to the momenta at any previous time t_0 together with the whole forces which have acted during the interval.

In the case in which no forces act on the system, except the mutual actions of the particles, we see that the momenta of all the particles of a system at any two times are equivalent; a result which has been already enunciated in Art. 40.

The two principles of the Conservation of Linear Momentum and Conservation of Areas may be enunciated as follows.

If the forces which act on a system be such that they have no component along a certain fixed straight line, then the motion is such that the linear momentum resolved along this line is constant.

If the forces be such that they have no moment about a certain fixed straight line, then the moment of the momentum or area conserved about this straight line is constant.

It is evident that these principles are only particular cases of the results proved in Art. 46, Chapter II.

130. Ex. 1. Suppose that a single particle m describes an orbit about a centre of force O. Let v, v' be its velocities at any two points P, P' of its course. Then mv' supposed to act along the tangent at P' if reversed would be in equilibrium with mv acting along the tangent at P together with the whole central force from P to P'. If p, p' be the lengths of the perpendiculars from O on the tangents at P, P', we have by taking moments about O, $vp = v'p'$, and hence vp is constant throughout the motion. Also if the tangents meet in T, the whole central force expended must act along the line TO, and may be found in terms of v, v' by the rules for compounding velocities.

131. Ex. 2. Suppose three particles to start from rest attracting each other, but under the action of no external forces. Then the momenta of the three particles at any instant are together equivalent to the three initial momenta and are therefore in equilibrium. Hence at any instant the tangents to their paths must meet in a point, and if parallels to their directions of motion be drawn so as to form a triangle, the momenta of the several particles are as the sides of that triangle.

If there are n particles it may be shown in the same way that the n forces represented by mv, $m'v'$, &c. are in equilibrium, and if parallels be drawn in the directions of motion and proportional to the momenta of the particles beginning at any point, they will form a closed polygon.

If F, F', F'' be the resultant attraction on the three particles, the lines of action of F, F', F'' also meet in a point. For let X, Y, Z be the actions between the particles $m'm''$, $m''m$, mm', taken in order. Then F is the resultant of $-Y$ and Z; F' of X and $-Z$; F'' of Y and $-X$. Hence the three forces F, F', F'' are in equilibrium*, and therefore their lines of action must meet in a point O'. Also the magnitude of each is proportional to the sine of the angle between the directions of the other two. This point is not generally fixed, and does not coincide with O.

If the law of attraction be proportional to the distance, the two points O, O' coincide with the centre of gravity G, and are fixed in space throughout the motion. For it is a known proposition in Statics that with this law of attraction, the whole attraction of a system of particles on one of the particles is the same as if the whole system were collected at its centre of gravity. Hence O' coincides with G. Also, since each particle starts from rest, the initial velocity of the centre of gravity is zero, and therefore, by Art. 46, G is a fixed point. Again, since each particle starts from rest and is urged towards a fixed point G, it will move in the straight line joining its initial position with G. Hence O coincides with G. When the law of attraction is proportional to the distance, it is proved in Dynamics of a Particle, that the time of reaching the centre of force from a position of rest is independent of the distance of that position of rest. Hence all the particles of the system will reach G at the same time, and meet there. If Σm be the sum of the masses, measured by their attractions in the usual manner, this time is known to be $\frac{1}{4}\frac{2\pi}{\sqrt{\Sigma m}}$.

132. Ex. 3. *Three particles whose masses are* m, m′, m″, *mutually attracting each other, are so projected that the triangle formed by joining their positions at any instant remains always similar to its original form. It is required to determine the conditions of projection.*

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* This proof is merely an amplification of the following. The three forces F, F', F'', being the internal re-actions of a system of three bodies, are in equilibrium by D'Alembert's Principle.

The centre of gravity will be either at rest or will move uniformly in a straight line. We may therefore consider the centre of gravity at rest and may afterwards generalise the conditions of projection by impressing on each particle an additional velocity parallel to the direction in which we wish the centre of gravity to move. Let O be the centre of gravity, P, P', P'' the positions of the particles at any time t. Then by the conditions of the question the lengths OP, OP', OP'' are always proportional, and their angular velocities about O are equal. Also the moment of the momenta of the system about O is always the same. Hence

$$mr^2n + m'r'^2n + m''r''^2n = \text{constant},$$

where r, r', r'' are the distances OP, OP', OP'', and n is their common angular velocity. Since the ratios $r : r' : r''$ are constants, it follows from this equation that mr^2n is constant, i.e. OP traces out equal areas in equal times. Hence by Newton, Section II, the resultant force on P tends towards O.

Let ρ, ρ', ρ'' be the sides $P'P''$, $P''P$, PP' of the triangle formed by the particles, and let the law of attraction be

$$\frac{\text{mass}}{(\text{dist.})^k}.$$

Then since the resultant attraction of m', m'' on m passes through O,

$$\frac{m'}{\rho''^k} \sin P'PO = \frac{m''}{\rho'^k} \sin P''PO,$$

but since O is the centre of gravity,

$$m'\rho'' \sin P'PO = m''\rho' \sin P''PO.$$

Hence either the three particles are in one straight line or $\rho''^{k+1} = \rho'^{k+1}$. If $k = -1$ the law of attraction is "as the distance." If k be not $= -1$, we have $\rho' = \rho''$, and the triangle must be equilateral.

Conversely, suppose the particles to be projected in directions making equal angles with their distances from the centre of gravity with velocities proportional to those distances, and suppose also the resultant attractions towards the centre of gravity to be proportional to those distances, then in all the three cases the same conditions will hold at the end of a time dt, and so on continually. The three particles will therefore describe similar orbits about the centre of gravity in a similar manner.

First, let us suppose that the three particles are to be in one straight line. To fix our ideas, let m' lie between m and m'', and O between m and m'. Then since the attraction on any particle must be proportional to the distance of that particle from O, the three attractions

$$\frac{m'}{(PP')^k}+\frac{m''}{(PP'')^k}, \qquad \frac{m''}{(P''P')^k}-\frac{m}{(PP')^k},$$

$$-\frac{m}{(PP'')^k}-\frac{m'}{(P'P'')^k},$$

must be proportional to OP, OP', OP''. Since $\Sigma mOP=0$, these two equations amount to but one on the whole. Let $z=\dfrac{P'P''}{PP'}$, so that $\dfrac{OP}{PP'}=\dfrac{m'+m''(1+z)}{m+m'+m''}$, $\dfrac{OP'}{PP'}=\dfrac{-m+m''z}{m+m'+m''}$.

Then we have

$$\left(m'+\frac{m''}{(1+z)^k}\right)(-m+m''z)=\left(\frac{m''}{z^k}-m\right)\{m'+m''(1+z)\},$$

which agrees with the result given by Laplace, by whom this problem was first considered.

In the case in which the attraction follows the law of nature $k=2$ and the equation becomes

$$mz^2\{(1+z)^3-1\}-m'(1+z)^2(1-z^3)-m''\{(1+z)^3-z^3\}=0.$$

This is an equation of the fifth degree, and it has therefore always one real root. The left side of the equation has opposite signs when $z=0$ and $z=\infty$, and hence this real root is positive. It is therefore always possible to project the three masses so that they shall remain in a straight line. Laplace remarks that if m be the sun, m' the earth, and m'' the moon, we have very nearly $z=\sqrt[3]{\dfrac{m+m''}{3m}}=\dfrac{1}{100}$. If then originally the earth and moon had been placed in the same straight line with the sun at distances from the sun proportional to 1 and $1+\frac{1}{100}$, and if their velocities had been initially parallel and proportional to those distances, the moon would have always been in opposition to the sun. The moon would have been too distant to have been in a state of continual eclipse, and thus would have been full every night. It has however been shown in Liouville's *Journal* that such a motion would be unstable.

The paths of the particles will be similar ellipses having the centre of gravity for a common focus.

Secondly. Let us suppose that the law of attraction is "as the distance." In this case the attraction on each particle is the same as if all the three particles were collected at the centre of gravity. Each particle will describe an ellipse having this point for centre in the same time. The necessary conditions of projection are that the velocities of projection should be proportional to the initial distances from the centre of gravity, and the directions of projection should make equal angles with those distances.

Thirdly. Let us suppose the particles to be at the angular points of an equilateral triangle. The resultant force on the particle m is

$$\frac{m'}{\rho''^k}\cos P'PO+\frac{m''}{\rho'^k}\cos P''PO.$$

The condition that the forces on the particles should be proportional to their distances from O shows that the ratio of this force to the distance OP is the same for all the particles. Since

$$m'\rho''\cos P'PO+m''\rho'\cos P''PO=(m+m'+m'')\,OP,$$

it is clear that the condition is initially satisfied when $\rho=\rho'=\rho''$. Hence, by the same reasoning as before, if the particles be projected with equal velocities in directions making equal angles with OP, OP', OP'' respectively, they will always remain at the angular points of an equilateral triangle.

133. When the system under consideration consists of rigid bodies we must use the results of Art. 42 to find the

resolved part of the momentum in any direction. The moment of the momentum about any straight line may also be found by Art. 43 in Chap. II. combined with Art. 69 in Chap. IV, if the motion be in two dimensions, or Art. 118 in Chap. V, if the motion be in three dimensions.

134. Ex. 4. *A disc of any form is moving in its own plane in any manner. Suddenly any point* O *in the disc is fixed, find the angular velocity of the disc about* O.

Let us suppose that just before O became fixed the centre of gravity G was moving with velocity V, and that p is the length of the perpendicular from O on the direction of motion. Also let ω be the angular velocity of the body about its centre of gravity. Just after O has become fixed, let the body be turning about O with angular velocity ω'. Let Mk^2 be the moment of inertia of the disc about the centre of gravity, and let $OG=r$.

The change in the motion of the disc is produced by impulsive forces acting at O during a short time t_1-t_0. These forces have no moment about O. Hence the moment of the momentum about O is the same just after and just before the impact. Just before O became fixed, the moment of the momentum about G was $Mk^2\omega$ (Art. 69), and the moment of the momentum of the whole mass collected at G was MVp. Hence the whole moment of the momentum about O is the sum of these two (Art. 43). Just after O has become fixed the body is turning about O, hence by Art. 69 the moment of the momentum about O is $M(k^2+r^2)\,\omega'$. Equating these we have

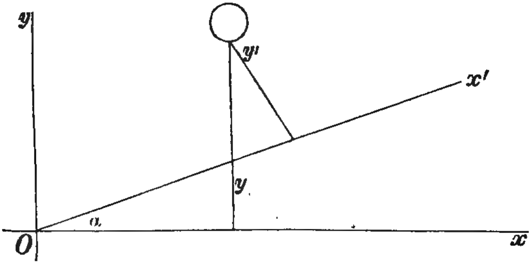
$$M(k^2+r^2)\,\omega' = Mk^2\omega + MVp;$$

$$\therefore\ \omega' = \frac{k^2\omega + Vp}{k^2+r^2}.$$

135. Ex. 5. *A disc of any form is turning about an axis* Ox *situated in its own plane with an angular velocity* ω. *Suddenly this axis is let free and another axis* Ox′, *also situated in the plane of the disc, becomes fixed, it is required to find the new angular velocity* ω' *about* Ox′.

The change in the motion of the disc is caused by the action of the impulsive forces due to the sudden fixing of the axis Ox'. These act at points situated in Ox' and have no moment about Ox'. Hence the moment of the momentum about Ox' is the same just before and just after Ox' is fixed.

Let $d\sigma$ be any element of the area of the disc; y, y' its distances from Ox, Ox'. Then $y\omega$, $y'\omega'$ are the velocities of $d\sigma$ just before and just after the impact. The moments of the momentum about Ox' just before and just after are therefore $yy'\omega d\sigma$ and $y'^2\omega' d\sigma$. Summing these for the whole area of the disc, we have



$$\omega'\Sigma y'^2 d\sigma = \omega \Sigma yy' d\sigma \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

First, let Ox, Ox' be parallel, so that the point O is at infinity. Let h be the distance between the axes, then $y' = y - h$. Hence we have

$$\omega'\Sigma y'^2 d\sigma = \omega \{\Sigma y^2 d\sigma - h\Sigma y d\sigma\}.$$

Let A, A' be the moments of inertia of the disc about Ox, Ox' respectively, $\bar{y}$ the distance of the centre of gravity from Ox, M the mass of the disc. Then we have

$$A'\omega' = \omega (A - Mh\bar{y}).$$

Secondly, let Ox, Ox' not be parallel. Let O be the origin and the angle $xOx' = \alpha$, then

$$y' = y \cos \alpha - x \sin \alpha.$$

Let F be the product of inertia of the disc about Ox, Oy where Oy is perpendicular to Ox. Then by substitution in (1) we have

$$A'\omega' = \omega\,(A\cos\alpha - F\sin\alpha).$$

136. *A rigid body is moving freely in space in a known manner. Suddenly either a straight line or a point in the body becomes fixed. To determine the initial subsequent motion.*

This proposition will include the last two examples as particular cases. It is obvious that all the impulsive actions on the body pass through the fixed straight line or the fixed point. Hence the moments of the momentum of the body about the fixed axis in the first case or about any axis through the fixed point in the second case are unaltered by the impulsive forces.

First. Let a straight line suddenly become fixed. Let it be taken as the axis of z.

Let MK^2 be the moment of inertia of the body about the axis of z, and Ω the angular velocity after the straight line has become fixed. Suppose that the body when moving freely was turning with angular velocities ω_x, ω_y, ω_z about three straight lines Gx', Gy', Gz' through the centre of gravity parallel to the axes of co-ordinates. And let the co-ordinates of the centre of gravity be $\bar{x}$, $\bar{y}$, $\bar{z}$.

Then

$$C'\omega_z - (\Sigma mz'x',)\,\omega_x - (\Sigma mz'y')\,\omega_y + M\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right)$$
$$= MK^2.\Omega,$$

where C' is the moment of inertia of the body about Gz', and $\Sigma mz'x'$, $\Sigma mz'y'$ are calculated with reference to the axes Gx', Gy', Gz'.

Secondly. Let a point O in the moving body be suddenly fixed in space. Take any three rectangular axes Ox, Oy, Oz, and three parallel axes Gx', Gy', Gz' through the centre of gravity G. Let ω_x, ω_y, ω_z be the known angular velocities of the body about the axes Gx', Gy', Gz' before the point O became fixed, Ω_x, Ω_y, Ω_z the unknown angular velocities about Ox, Oy, Oz after O has become fixed.

Then, following the same notation as before, we have

$$A'\omega_x - (\Sigma m\, x'y')\,\omega_y - (\Sigma m\, x'z')\,\omega_z + \Sigma m \left(\bar{y}\frac{d\bar{z}}{dt} - \bar{z}\frac{d\bar{y}}{dt}\right)$$
$$= A\Omega_x - (\Sigma m\, xy)\,\Omega_y - (\Sigma m\, xz)\,\Omega_z.$$

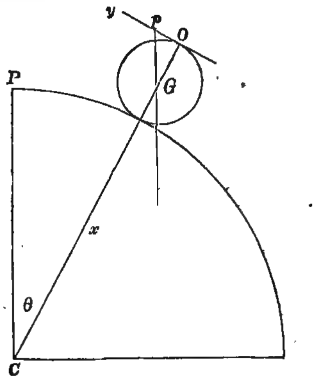
$$B'\omega_y - (\Sigma m\, y'z')\,\omega_z - (\Sigma m\, y'x')\,\omega_x + \Sigma m \left(\bar{z}\frac{d\bar{x}}{dt} - \bar{x}\frac{d\bar{z}}{dt}\right)$$
$$= B\Omega_y - (\Sigma m\, yz)\,\Omega_z - (\Sigma m\, yx)\,\Omega_x.$$

$$C'\omega_z - (\Sigma m\, z'x')\,\omega_x - (\Sigma m\, z'y')\,\omega_y + \Sigma m \left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right)$$
$$= C\Omega_z - (\Sigma m\, zx)\,\Omega_x - (\Sigma m\, zy)\,\Omega_y.$$

These equations determine Ω_x, Ω_y, Ω_z. It is very obvious that they may be greatly simplified by so choosing the axes that one of the two sets Ox, Oy, Oz or Gx', Gy', Gz' may be a set of principal axes.

137. Ex. *A sphere in co-latitude θ is hung up by a point O in its surface in equilibrium under the action of gravity. Suddenly the rotation of the earth is stopped, it is required to determine the motion of the sphere.*

Let G be the centre of the sphere, O its point of suspension, and a its radius. Let C be the centre of the earth. Let us suppose the figure so drawn that the sphere is moving away from the observer.



Let $\omega =$ angular velocity of the earth, then if $CG = \mu a$, the sphere is turning about an axis Gp parallel to CP, the axis of the earth with angular velocity ω, while the centre of gravity is moving with velocity $\mu a \sin\theta \,.\, \omega$.

Let OC, Op, and the perpendicular to the plane of OC, Op be taken as the axes of x, y, z respectively, and let Ω_x, Ω_y, Ω_z be the angular velocities about them just after the rotation of the earth is stopped.

By Art. 136, the angular momenta about Ox, just before and just after the rotation was stopped, are equal to each other;

$$\therefore\ Mk^2 \omega \cos\theta = Mk^2 \Omega_x,$$

where Mk^2 is the moment of inertia of the sphere about a diameter.

Again, the angular momenta about Oy are equal to each other;

$$\therefore\ -Mk^2 \omega \sin\theta + M\mu a^2 \omega \sin\theta = M(k^2 + a^2)\,\Omega_y.$$

Lastly, the angular momenta about Oz are equal;

$$\therefore\ 0 = Mk^2 \Omega_z.$$

Solving these equations, we get

$$\Omega_y = \omega \sin\theta \frac{-k^2 + \mu a^2}{k^2 + a^2}$$

$$= \omega \sin\theta \frac{-2 + 5\mu}{7}.$$

But $\qquad \Omega_x = \omega \cos\theta.$

Adding together the squares of Ω_x, Ω_y, Ω_z we have

$$\Omega^2 = \omega^2 \left\{ \cos^2\theta + \left(\frac{-2 + 5\mu}{7} \right)^2 \sin^2\theta \right\},$$

where Ω is the angular velocity of the sphere about its instantaneous axis.

The Invariable Plane.

138. It is shown in Art. 40 of Chap. II. that all the momenta of the several particles of a system in motion, are together equivalent to a single resultant linear momentum at any assumed origin O, represented in direction and magnitude by a line OV, together with an angular momentum about some line passing through O, represented in direction and magnitude by a line OH. Let h_1, h_2, h_3 be the moments of the momenta of the particles about any rectangular axes Ox, Oy, Oz meeting in O, so that

$$h_1 = \Sigma m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right),$$

with similar expressions for h_2, h_3, and let

$$h^2 = h_1^2 + h_2^2 + h_3^2.$$

Then the direction cosines of OH are $\frac{h_1}{h}$, $\frac{h_2}{h}$, $\frac{h_3}{h}$ and the angular momentum itself is represented by h.

If no external forces act on the system then h_1, h_2, h_3, are constant throughout the motion, Art. 40, hence OH is fixed in direction and magnitude. It is therefore called the invariable line at O, and a plane perpendicular to OH is called the invariable plane at O.

This plane may be conveniently used as a plane of reference by which the positions of the bodies at any time may be determined. It may be deduced at any future time from the motions of the bodies forming the system, just as the position of the centre of gravity may be found, so that the knowledge of its position cannot be lost. In the case of the solar system, we may, for instance, refer the inclinations of the planes of the orbits to this fixed plane.

139. The position of the invariable plane at the centre of gravity of the solar system may be found in the following manner. Let the system be referred to any rectangular axes meeting in the centre of gravity. Let ω be the angular

velocity of any planet about its axis of rotation. Let Mk^2 be its moment of inertia about that axis and (α, β, γ) the direction angles of that axis. The axis of revolution and two perpendicular axes form a system of principal axes at the centre of gravity. The angular momentum about the axis of revolution is $Mk^2\omega$, and hence the angular momentum about an axis parallel to the axis of z is $Mk^2\omega\cos\gamma$. The moment of the momentum of the whole mass collected at the centre of gravity about the axis of z is

$$M\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right),$$

hence we have

$$h_3 = Mk^2\omega\cos\gamma + M\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right).$$

The values of h_1, h_2, may be found in a similar manner. The position of the invariable plane is then known.

If the planets forming the system be regarded as spheres so that their mutual attractions act along the straight lines joining their centres, the motions about their centres will be always the same. In this case we might regard them as a system of mutually attracting points. This is what is usually done, and the terms depending on the rotations of the planets in the preceding values of h_1, h_2, h_3, are then omitted. The plane thus found is normal to the axis of the momentum-couple due to the motions of the several masses collected at their centres of gravity. It is not therefore strictly fixed in space, because the attractions of the planets do not strictly act along the straight lines joining their centres of gravity. But in any long time the changes produced in the rotations of the planets are so small compared with the other terms retained in the expressions for h_1, h_2, h_3, that the plane may be regarded as fixed.

140. If any straight line OL be drawn through O making an angle θ with the invariable line OH at O, the angular momentum about OL is $h\cos\theta$. For the axis of the resultant momentum-couple is OH, and the resolved part about OL is therefore $OH\cos\theta$. Hence the invariable line at O

may also be defined as that axis through O about which the moment of the momentum is greatest.

141. At different points of the system the position of the invariable line is different. But the rules by which they are connected are the same as those which connect the axes of the resultant couple of a system of forces when the origin of reference is varied. These have been already stated in Art. 104 of Chap. v., and it is unnecessary here to do more than generally to refer to them.

Impulsive Forces in Three Dimensions.

142. *To determine the general equations of motion of a body about a fixed point under the action of given impulses.*

Let the fixed point be taken as the origin, and let ω_x', ω_y', ω_z' be the angular velocities *generated by the impulses* about any rectangular axes of co-ordinates. By D'Alembert's Principle, see Art. 50, the difference between the moments of the momenta of the particles of the system just before and just after the action of the impulses is equal to the moment of the impulses. Hence by Art. 118 of Chap. v.,

$$\left.\begin{aligned} A\omega_x' - (\Sigma mxy)\,\omega_y' - (\Sigma mxz)\,\omega_z' &= L \\ B\omega_y' - (\Sigma myz)\,\omega_z' - (\Sigma myx)\,\omega_x' &= M \\ C\omega_z' - (\Sigma mzx)\,\omega_x' - (\Sigma mzy)\,\omega_y' &= N \end{aligned}\right\} \ldots\ldots(1),$$

where L, M, N are the moments of the impulsive forces about the axes.

These three equations will suffice to determine the values of ω_x', ω_y', ω_z'. These being added to the angular velocities before the impulse, the initial motion of the body after the impulse is found.

It is to be observed that these equations leave the axes of reference undetermined. They should be so chosen that the values of A, Σmxy, &c. may be most easily found. If the positions of the principal axes at the fixed point are known they will in general be found the most suitable.

In that case the equations reduce to the simple form

$$\left.\begin{aligned} A\omega_x' &= L \\ B\omega_y' &= M \\ C\omega_z' &= N \end{aligned}\right\} \quad \text{......................}(2).$$

The values of ω_x', ω_y', ω_z' being known, we can find the pressures on the fixed point. For by D'Alembert's Principle the change in the linear momentum of the body in any direction is equal to the resolved part of the impulsive forces. Hence if F, G, H, be the pressures of the fixed point on the body

$$\left.\begin{aligned} \Sigma X + F &= M . \frac{d\bar{x}}{dt} \text{ by Art. 49} \\ &= M(\omega_y'\bar{z} - \omega_z'\bar{y}) \text{ by Art. 105} \\ \Sigma Y + G &= M(\omega_z'\bar{x} - \omega_x'\bar{y}) \\ \Sigma Z + H &= M(\omega_x'\bar{y} - \omega_y'\bar{x}) \end{aligned}\right\} \text{...}(3).$$

143. If the body be free, the motion round the centre of gravity will be the same as if that point were fixed. Hence the axes being any three straight lines meeting at the centre of gravity, the angular velocities of the body may still be found by the equations (1) or (2). The motion of the centre of gravity may be found from (3). Let $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{u}'$, $\bar{v}'$, $\bar{w}'$ be the resolved parts of the velocities of the centre of gravity before and after the impulses, and let M be the whole mass. Then these equations become

$$\bar{u}' - \bar{u} = \frac{\Sigma X}{M},$$

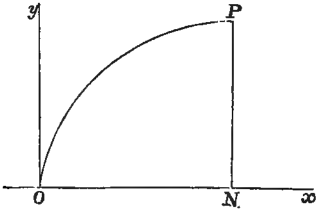
$$\bar{v}' - \bar{v} = \frac{\Sigma Y}{M},$$

$$\bar{w}' - \bar{w} = \frac{\Sigma Z}{M}.$$

On the general subject of impacts the reader may consult an Article in Liouville's *Journal*, Vol. XIV.

144. Ex. *A portion of a parabola bounded by an ordinate* PN, *the axis* ON, *and the curve* OP, *has its vertex* O *fixed. A blow* P *is given to it perpendicular to its plane at the other extremity of the curved boundary. Supposing it at rest before the blow, find the initial motion.*

Let the equation to the parabola be $y^2 = 4ax$.



Then $$\Sigma mxz = 0, \quad \Sigma myz = 0.$$

Let μ be the mass of a unit of area, and M the whole mass, and $ON = c$,

$$\text{then } \Sigma mxy = \mu \iint xy\, dx\, dy = \mu \int x \frac{y^2}{2}\, dx \begin{cases} y = 0 \\ y = y \end{cases}$$

$$= 2\mu \int ax^2\, dx = \frac{2}{3} \mu ac^3 \qquad \begin{cases} x = 0 \\ x = c \end{cases}$$

$$= M \frac{\sqrt{ac^3}}{2}.$$

Also $$A = \frac{1}{3} \mu \int_0^c y^3\, dx = \frac{16}{15} \mu a^{\frac{3}{2}} c^{\frac{5}{2}} = M . \frac{4ac}{5},$$

$$B = \mu \int_0^c x^2 y\, dx = \frac{4}{7} \mu a^{\frac{1}{2}} c^{\frac{7}{2}} = M \frac{3c^2}{7};$$

and the equations are

$$\left.\begin{aligned} A\omega_x - \frac{2}{3} \mu ac^3 \omega_y &= -P . 2\sqrt{ac} \\ B\omega_y - \frac{2}{3} \mu ac^3 \omega_x &= Pc \\ (A + B)\, \omega_z &= 0 \end{aligned}\right\},$$

whence $\omega_x \omega_y$ may be easily found.

And the pressures on the fixed point may be found from the equations

$$F=0, \quad G=0,$$

$$P+H=M(\omega_x\bar{y}-\omega_y\bar{x}).$$

The axis about which the body *begins* to turn makes an angle with Ox, whose tangent is

$$\tan\theta=\frac{\omega_y}{\omega_x},$$

and the initial angular velocity $=\sqrt{\omega_x^2+\omega_y^2}$.

But the body will not *continue* to rotate about this axis unless it be also a principal axis.

145. Prop. *To show that we may take moments about the initial axis of rotation as if it were a fixed axis.*

Let l, m, n be the direction cosines of the initial axis of rotation; let I be the moment of inertia, and Ω the angular velocity of the body about it. Let G' be the moment of the forces about the same axis. Then

$$\omega_x=l\Omega, \quad \omega_y=m\Omega, \quad \omega_z=n\Omega.$$

Hence the equations of motion become

$$Al\Omega=L, \quad Bm\Omega=M, \quad Cn\Omega=N.$$

Multiplying these by l, m, n, and adding, we get

$$(Al^2+Bm^2+Cn^2)\,\Omega=Ll+Mm+Nn,$$

or

$$I\,.\,\Omega=G'.$$

146. Prop. *A body at rest being acted on by any impulses, it is required to find the condition that the resulting motion may be one of rotation only.*

It has been proved that the motion of any body can always be represented by a motion of rotation about some axis, and a motion of translation in the direction of the axis. The condition that the motion may be one of rotation only is, by Art. 100,

$$\bar{u}'\omega_x'+\bar{v}'\omega_y'+\bar{w}'\omega_z'=0.$$

But substituting for $\bar{u}'$, &c., ω_x', &c., their values in terms of the forces given in Arts. 142 and 143, this becomes

$$\frac{L\Sigma X}{A} + \frac{M\Sigma Y}{B} + \frac{N\Sigma Z}{C} = 0.$$

This condition is necessary, but not sufficient. It is also necessary that L, M, N do not all vanish.

147. Ex. *A cricket-ball is set rotating about a horizontal axis in the vertical plane of motion with an angular velocity* ω_1. *Supposing that when it strikes the ground the centre is moving with velocity* V *in a direction making an angle* α *with the horizon, find the subsequent motion.*

Take the normal to the ground at the point where the ball touches as the axis of z, and let the axis of x be in the plane of motion of the ball before impact. Let ω_1', ω_2', ω_3' be the angular velocities just after impact of the ball about the diameters parallel to the axes. Let u', v', w' be the resolved parts of the velocities of the centre. Let Z be the normal reaction of the ground, X, Y the frictional impulsive actions, estimated positively in the positive directions of the axes. Let m be the mass of the ball.

The equations of motion are

$$\left.\begin{aligned} \omega_1' - \omega_1 &= \frac{Ya}{mk^2} \\ \omega_2' &= -\frac{Xa}{mk^2} \\ \omega_3' &= 0 \end{aligned}\right\} \text{......................} (1),$$

$$\left.\begin{aligned} u' - V\cos\alpha &= \frac{X}{m} \\ v' &= \frac{Y}{m} \\ w' + V\sin\alpha &= \frac{Z}{m} \end{aligned}\right\} \text{......................} (2).$$

The geometrical equations are

$$\left.\begin{aligned} u' - a\omega_2' &= 0 \\ v' + a\omega_1' &= 0 \\ w' &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

These hold at the moment of greatest compression. Solving these, we get

$$X = -\frac{k^2}{a^2+k^2} \cdot mV\cos\alpha,$$

$$Y = -\frac{k^2}{a^2+k^2} \cdot ma\omega_1,$$

$$Z = mV\sin\alpha.$$

Let e' be the frictional and e the normal elasticity. Then these values of X, Y, Z must be multiplied respectively by $1+e'$, $1+e'$, and $1+e$.

By substituting in (1) and (2), we get

$$\omega_1' = \omega_1 \frac{k^2 - a^2e'}{k^2+a^2}, \qquad u' = V\cos\alpha\,\frac{a^2 - k^2e'}{k^2+a^2},$$

$$\omega_2' = \frac{Va\cos\alpha}{k^2+a^2}(1+e'), \qquad v' = -\frac{a\omega_1k^2}{k^2+a^2}(1+e'),$$

$$\omega_3' = 0, \qquad w' = eV\sin\alpha.$$

It appears from these equations that unless $\omega_1 = 0$ the ball will not move in the same vertical plane after impact as before. Let θ be the angle made by the two vertical planes of motion. Then $\tan\theta = \frac{v'}{u'}$, and after putting $k^2 = \frac{2}{5}a^2$, we get

$$\tan\theta = -\frac{2a\omega_1}{V\cos\alpha} \cdot \frac{1+e'}{5-2e'}.$$

148. If the ground be imperfectly rough with a coefficient of friction μ, it is possible that the friction may not be sufficient to prevent sliding. In this case the magnitude of the friction at any instant during the impact is known, being proportional to the normal pressure. Its direction is opposite

to that of the motion of the point of the sphere in contact with the ground, and therefore may not remain the same during the whole impact. Let X, Y be the whole resolved parts of the frictions (measured by the momenta they would generate) from the beginning of the impact up to a time t. Then $\dfrac{dX}{dt}$ and $\dfrac{dY}{dt}$ are the moving forces at that instant acting on the ball. The resultant of these must be opposite to the direction of motion of the point of contact of the ball with the ground, hence

$$\frac{dY}{dX} = \frac{v' + a\omega_1'}{u' - a\omega_2'} \text{ ....} \quad (4),$$

also
$$\left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2 = \mu^2 \left(\frac{dZ}{dt}\right)^2 \text{ ....} (5),$$

supposing the whole friction to be called into play.

From these we get by equations (1) and (2),

$$\frac{dY}{dX} = \frac{Y\left(1 + \frac{a^2}{k^2}\right) + ma\omega_1}{X\left(1 + \frac{a^2}{k^2}\right) + mV\cos\alpha}.$$

Collecting all the X's on one side of the equation and all the Y's on the other and integrating, we get

$$Y\left(1 + \frac{a^2}{k^2}\right) + ma\omega_1 = L\left\{X\left(1 + \frac{a^2}{k^2}\right) + mV\cos\alpha\right\},$$

where L is some constant. Here X and Y are the accumulations of momenta from the beginning of the impact, hence when $t = 0$, X and Y both vanish; therefore $a\omega_1 = LV\cos\alpha$. Substituting for L we have

$$\frac{Y}{X} = \frac{a\omega_1}{V\cos\alpha} \text{ ....} \quad (6).$$

Hence the direction of the friction is constant throughout the impulse.

Substituting in equation (5) and integrating, we have

$$X\sqrt{1+\frac{a^2\omega_1^2}{V^2\cos^2\alpha}}=\mu Z.$$

But since $w'=0$ when the impact has terminated, we have by (2) $Z=mV\sin\alpha$. Hence the whole value of X from the beginning to the termination of the impact has been found. The value of Y is then known by (6). Substituting these values of X and Y in equations (1) and (2) the initial motion of the sphere is found.

149. In the same way we may show that if two imperfectly rough spheres impinge on each other, the direction of friction is the same throughout the impact and opposite to the initial direction of relative motion. If a solid of revolution impinge with its vertex on an imperfectly rough horizontal plane the direction of friction is constant. If two solid ellipsoids, having their principal diameters parallel, impinge on each other so as to touch each other at their vertices, and if the initial direction of relative motion be along a common principal axis of both ellipsoids at the point of contact, the direction of friction will in this case also be constant.

EXAMPLES.

1. A cone revolves round its axis with a known angular velocity. The altitude begins to diminish and the angle to increase, the volume being constant. Show that the angular velocity is proportional to the altitude.

2. A circular disc is revolving in its own plane about its centre; if a point in the circumference become fixed, find the new angular velocity.

3. A uniform rod of length $2a$ lying on a smooth horizontal plane passes through a ring which permits the rod to rotate freely in the horizontal plane. The middle point of the rod being indefinitely near the ring any angular velocity is impressed on it, show that when it leaves the ring the

radius vector of the middle point will have swept out an area equal to $\frac{a^2}{6}$.

4. A spherical shell contains an interior spherical cavity whose diameter is a radius of the shell, and which is filled with fluid of the same density as the matter of the shell. A tangential impulse is given to the shell at the extremity of the diameter of the cavity, show that the angular velocity communicated to the system is $\frac{32}{31}$ times what it would have been if the fluid had been solid and rigidly connected with the shell.

5. An elliptic lamina is rotating about its centre on a smooth horizontal table. If ω_1, ω_2, ω_3 be its angular velocities respectively when the extremities of its major axis, its focus, and the extremity of the minor axis become fixed, prove

$$\frac{7}{\omega_1} = \frac{6}{\omega_2} + \frac{5}{\omega_3}.$$

6. A rigid body moveable about a fixed point O at which the principal moments are A, B, C is struck by a blow of given magnitude at a given point. If the angular velocity thus impressed on the body be the greatest possible, prove that (a, b, c) being the co-ordinates of the given point referred to the principal axes at O, and (l, m, n) the direction cosines of the blow, then

$$al + bm + cn = 0,$$

$$\frac{a}{l}\left(\frac{1}{B^2} - \frac{1}{C^2}\right) + \frac{b}{m}\left(\frac{1}{C^2} - \frac{1}{A^2}\right) + \frac{c}{n}\left(\frac{1}{A^2} - \frac{1}{B^2}\right) = 0.$$

7. Any triangular lamina ABC has the angular point C fixed and is capable of free motion about it. A blow is struck at B perpendicular to the plane of the triangle. Show that the initial axis of rotation is that trisector of the side AB which is furthest from B.

8. A body is turning about an axis through its centre of gravity, a point in the body becomes suddenly fixed. If

the new instantaneous axis be a principal axis with respect to the point, show that the locus of the point is a rectangular hyperbola.

9. A cube is rotating with angular velocity ω about a diagonal, when one of its edges which does not meet the diagonal suddenly becomes fixed. Show that the angular velocity about this edge as axis $= \dfrac{\omega}{4\sqrt{3}}$.

10. Two masses m, m' are connected by a fine smooth string which passes round a right circular cylinder of radius a. The two particles are in motion in one plane under no impressed forces, show that if A be the sum of the absolute areas swept out in a time t by the two unwrapped portions of the string

$$\frac{d^2A}{dt^2} = \frac{1}{2} a \left(\frac{1}{m} + \frac{1}{m'}\right) T,$$

T being the tension of the string at any time.

11. A piece of wire in the form of a circle lies at rest with its plane in contact with a smooth horizontal table, when an insect on it suddenly starts walking along the arc with uniform relative velocity. Show that the wire revolves round its centre with uniform angular velocity while that centre describes a circle in space with uniform angular velocity.

12. A uniform circular wire of radius a moveable about a fixed point in its circumference, lies on a smooth horizontal plane. An insect of mass equal to that of the wire crawls along it, starting from the extremity of the diameter opposite to the fixed point, its velocity relative to the wire being uniform and equal to V. Prove that after a time t the wire will have turned through an angle

$$\frac{Vt}{2a} = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan\frac{Vt}{2a}\right).$$

13. A small insect moves along a uniform bar of mass equal to itself, and length $2a$, the extremities of which are constrained to remain on the circumference of a fixed circle,

whose radius is $\frac{2a}{\sqrt{3}}$. Supposing the insect to start from the middle point of the bar, and its velocity relatively to the bar to be uniform and equal to V; prove that the bar in time t will turn through an angle

$$\frac{1}{\sqrt{3}}\tan^{-1}\frac{Vt}{a}.$$

14. A uniform circular disc moveable about its centre in its own plane (which is horizontal) has a fine groove in it cut along a radius, and is set rotating with angular velocity ω. A small rocket whose weight is $\frac{1}{n}$ the weight of the disc is placed at the inner extremity of the groove and discharged; and when it has left the groove, the same is done with another equal rocket, and so on. Find the angular velocity after n of these operations, and if n be indefinitely increased, show that the limiting value of the same is ωe^{-2}.

15. A rigid body is rotating about an axis through its centre of gravity, when a certain point of the body becomes suddenly fixed, the axis being simultaneously set free; find the equations of the new instantaneous axis; and prove that, if it be parallel to the originally fixed axis, the point must lie in the line represented by the equations

$$a^2lx + b^2my + c^2nz = 0,$$

$$(b^2 - c^2)\frac{x}{l} + (c^2 - a^2)\frac{y}{m} + (a^2 - b^2)\frac{z}{n} = 0;$$

the principal axes through the centre of gravity being taken as axes of co-ordinates, a, b, c the radii of gyration about these lines, and l, m, n the direction cosines of the originally fixed axis referred to them.

CHAPTER VII.

VIS VIVA.

150. LET x, y, z be the co-ordinates of a particle m of a system in motion at any given instant; and let X, Y, Z be the accelerating forces acting on the particle resolved in the directions of the axes. Then, the summation being extended throughout the system, the function

$$\Sigma m (Xdx + Ydy + Zdz)$$

will in general be a complete differential of some quantity U. This quantity U, when it exists, is called *the force function.*

This name has been given to it by Sir W. Hamilton and Jacobi independently of each other.

151. PROP. I. *To prove that there will be a force function first, when the forces tend to fixed centres at finite distances, and are functions of the distances from those centres; and secondly, when the forces are due to the mutual attractions or repulsions of the particles of the system, and are functions of the distances between the attracting or repelling particles.*

Let $m\phi(r)$ be the action of any fixed centre of force on a particle m distant r, estimated in the direction in which r is measured, i.e. *from* the centre of force. Then all the forces due to the several centres of force are exactly equivalent to all their several components mX, mY, mZ. Hence, reversing the latter, there will be equilibrium. Therefore, by Virtual Velocities,

$$\Sigma m\phi(r)\, dr = \Sigma m (Xdx + Ydy + Zdz).$$

Thus the force function exists, and is equal to

$$\Sigma m \int \phi(r)\, dr.$$

If the law of attraction be the inverse-square $\phi(r) = -\frac{\mu}{r^2}$, and the integral becomes $\frac{\mu}{r}$. Thus the force function becomes what has been called in Statics the Potential.

In the same way it can be shown that there will be a force function when the forces are such as result from the attraction of the particles of the system on each other, provided the attractions are functions only of the distances between the particles.

Let $mm'\phi(r)$ be the action between two particles m, m', whose distance apart is r; and, as before, let this force be considered positive when repulsive. Then we have

$$\Sigma mm'\phi(r)\, dr = \Sigma m\, (Xdx + Ydy + Zdz),$$

and the force function is equal to

$$\Sigma mm' \int \phi(r)\, dr.$$

152. Prop. II. *If a system receive any small displacement* ds *parallel to a given straight line and an angular displacement* dθ *round that line, then the partial differential coefficients* $\frac{\mathrm{dU}}{\mathrm{ds}}$ *and* $\frac{\mathrm{dU}}{\mathrm{d}\theta}$ *represent respectively the resolved part of all the forces along the line and the moment of the forces about it.*

Since dU is the sum of the virtual moments of all the forces due to any displacement, it is independent of any particular co-ordinate axes. Let the straight line along which ds is measured be taken as the axis of z.

Taking the same notation as before,

$$dU = \Sigma m\, (Xdx + Ydy + Zdz).$$

But $dx = 0$, $dy = 0$, and $dz = ds$, hence we have

$$dU = ds \, . \, \Sigma mZ;$$

$$\therefore \frac{dU}{ds} = \Sigma mZ.$$

Here dU means the change produced in U by the single displacement of the system, taken as one body, parallel to the given straight line, through a space ds.

Again, the moment of all the forces about the axis of z is

$$\Sigma m\,(xY - yZ),$$

but by Art. 105, $dx = -y\,d\theta$, and $dy = x\,d\theta$, and $dz = 0$, hence the above moment is

$$= \Sigma m \,.\, \frac{Ydy + Xdx + Zdz}{d\theta}$$

$$= \frac{dU}{d\theta}\,.$$

Here dU is the change produced in U by the single rotation of the system, taken as one body, round the given axis, through an angle $d\theta$.

153. Def. The *Vis Viva* of a particle is the product of its mass into the square of its velocity.

If a system be in motion under the action of finite forces, and if the geometrical relations of the parts of the system be expressed by equations which do not contain the time explicitly, the change in the vis viva of the system in passing from any one position to any other is equal to twice the corresponding change produced in the force function.

In determining the force function all forces may be admitted which would not appear in the equation of Virtual Velocities.

Let x, y, z be the co-ordinates of any particle m, and let X, Y, Z be the resolved parts in the directions of the axes of the impressed accelerating forces acting on the particle.

The effective forces acting on the particle m at any time t are

$$m\frac{d^2x}{dt^2}, \quad m\frac{d^2y}{dt^2}, \quad m\frac{d^2z}{dt^2}\,.$$

If the effective forces on all the particles be reversed, they will be in equilibrium with the whole group of impressed forces by Art. 37. Hence, by the principle of virtual velocities,

$$\Sigma m \left\{\left(X - \frac{d^2x}{dt^2}\right)\delta x + \left(Y - \frac{d^2y}{dt^2}\right)\delta y + \left(Z - \frac{d^2z}{dt^2}\right)\delta z\right\} = 0,$$

where δx, δy, δz are any small arbitrary displacements of the particle m consistent with the geometrical relations at the time t.

Now if the geometrical relations be expressed by equations which do not contain the time explicitly, the geometrical relations which hold at the time t will hold throughout the time δt; and therefore we can take the *arbitrary* displacements δx, δy, δz to be respectively equal to the *actual* displacements

$$\frac{dx}{dt}\delta t, \quad \frac{dy}{dt}\delta t, \quad \frac{dz}{dt}\delta t$$

of the particle in the time δt.

Making this substitution, the equation becomes

$$\Sigma m \left(\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt} + \frac{d^2z}{dt^2}\frac{dz}{dt}\right)$$

$$= \Sigma m \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right).$$

Integrating, we get

$$\Sigma m \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}$$

$$= C + 2\Sigma m \int (X dx + Y dy + Z dz),$$

where C is the constant to be determined by the initial conditions of motion.

Let v and v' be the velocities of the particle m at the times t and t'. Also let U, U' be the values of the force function for the system in the two positions which it has at the times t and t'. Then

$$\Sigma m v'^2 - \Sigma m v^2 = 2(U' - U).$$

154. The following illustration, taken from Poisson, may show more clearly why it is necessary that the geometrical relations should not contain the time explicitly. Let, for example,

$$\phi(x, y, z, t) = 0 \text{.......................}(1)$$

be any geometrical relation connecting the co-ordinates of the particle m. This may be regarded as the equation to a moving surface on which the particle is constrained to rest. The quantities δx, δy, δz are the projections on the axes of any arbitrary displacement of the particle m consistent with the geometrical relations which hold at the time t. They must therefore satisfy the equation

$$\frac{d\phi}{dx}\delta x + \frac{d\phi}{dy}\delta y + \frac{d\phi}{dz}\delta z = 0.$$

The quantities $\frac{dx}{dt}\delta t$, $\frac{dy}{dt}\delta t$, $\frac{dz}{dt}\delta t$ are the projections on the axes of the displacement of the particle due to its motion in the time δt. They must therefore satisfy the equation

$$\frac{d\phi}{dx}\frac{dx}{dt}\delta t + \frac{d\phi}{dy}\frac{dy}{dt}\delta t + \frac{d\phi}{dz}\frac{dz}{dt}\delta t + \frac{d\phi}{dt}\delta t = 0.$$

Hence unless $\frac{d\phi}{dt}$ is zero throughout the whole motion we cannot take δx, δy, δz to be respectively equal to

$$\frac{dx}{dt}\delta t, \quad \frac{dy}{dt}\delta t, \quad \frac{dz}{dt}\delta t.$$

The equation $\frac{d\phi}{dt} = 0$ expresses the condition that the geometrical equation (1) should not contain the time explicitly.

155. The equation of Virtual Velocities in Statics is known to contain in one formula all the conditions of equilibrium. In the same way the general equation

$$\Sigma m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right) = \Sigma m\,(X\delta x + Y\delta y + Z\delta z),$$

may be made to give all the equations of motion by properly choosing the arbitrary displacements δx, δy, δz. In the last Article we made one choice of these displacements and thus obtained an equation in an integrable form.

If we give the whole system a displacement parallel to the axis of z we have $\delta x = 0$, $\delta y = 0$, and δz is arbitrary. The equation then becomes

$$\Sigma m \frac{d^2 z}{dt^2} = \Sigma m Z,$$

which represents any one of the three first general equations of motion in Art. 39.

If we give the whole system a displacement round the axis of z through an angle $\delta\theta$, we have $\delta x = -y\delta\theta$, $\delta y = x\delta\theta$, $\delta z = 0$. The equation then becomes

$$\Sigma m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = \Sigma m (xY - yX),$$

which represents any one of the three last general equations of motion in Art. 39.

156. Let a force P act on a particle which describes the elementary arc ds in the time dt. Let dp be the projection of ds on the line of action of P estimated positive in the direction in which P acts. Then Pdp is called the "*work*" done by the force P in the time dt. If a force act continuously on a particle during a time T, then the work will clearly be represented by $\int_0^T \frac{Pdp}{dt}\, dt$. Let X, Y, Z be the resolved parts in the directions of the axes of the accelerating forces which act on any particle m. Then the whole "work" done, while the system is moving from one position to another, is

$$W = \Sigma m \int (Xdx + Ydy + Zdz).$$

Let U, U' be the values of the force function of the system corresponding to the two positions of the system. Then

$$W = U' - U.$$

By the principle of vis viva we have

$$\Sigma mv'^2 - \Sigma mv^2 = 2(U' - U).$$

Therefore the change in the vis viva of a system in moving from one position to another is equal to twice the "work" done by the forces during the motion.

We may use the term "force function" to indicate the indefinite integral U and the term "work" to indicate the same taken between any given limits.

Suppose a weight mg to be placed at any height above the surface of the earth. Then as it falls through a height z it may be made by any machine to do "work" measured by mgz. This follows from the principle of Virtual Velocities. If it work no machine it has acquired a vis viva, the half of which is equal to $m\frac{v^2}{2}$ or mgz. If it fall through a second space z', it may be made either to do more work by the machine or to generate an additional semi vis viva equal to mgz'. And so on until it has fallen as far as the circumstances of the case permit. After this no more work can be done by the weight until it has been lifted up again. If it be raised up to the height h by being projected with the proper velocity v, an amount of semi vis viva has disappeared measured by $m\frac{v^2}{2} = mgh$. Or if the weight be wound up by some machine this measures the work done by the machine. In any case the weight has now acquired a capacity to do work measured by mgh. Thus we see that vis viva may be changed into a capacity to do work, and conversely.

Let us apply this reasoning to the general case of a system of particles subject to no external forces but only to the mutual action of its several parts. Let us suppose the system to have moved from its arrangement at the time t_0, which we might take at the beginning of the motion, to its arrangement at the time t. Let t_1 be any other time greater than t_0 and such that we need only consider the motion between the two times t_0 and t_1. If necessary t_1 may be regarded as infinitely great. Let W_0 be the work done by the mutual actions while the system moves from the arrangement at the time t_0 to that at the time t_1, W the work done while it moves from the arrangement at the time t to that at the time t_1. Let T_0 be respectively the actual semi vis viva at the time t_0, T that at the time t. Then by the equation of vis viva, we have

$$T - T_0 = W_0 - W,$$

or

$$T + W = T_0 + W_0.$$

Hence it appears that the quantity $T + W$ is constant throughout the motion. To express this the term "*energy*" has been introduced by W. J. M. Rankine, C.E.* Actual energy at the time t is the semi vis viva of the system at that instant. Potential energy is the remaining work, whether positive or negative, which the forces can be made to do by moving the system from its arrangement at the time t to the arrangement at some fixed future time. The principle asserts that the sum of these two is constant throughout the motion.

If any external forces act on the system the total energy both actual and potential has been increased positively or negatively by a quantity equal to the work done by the external forces, as the system is moved from its position at the time t_0 to its position at the time t.

157. In applying the principle of vis viva to any actual cases, it will be important to know beforehand what forces and internal reactions may be disregarded in forming the equation. The general rule is that all forces may be neglected which do not appear in the equation of Virtual Velocities. These forces may be enumerated as follows:

I. Those reactions whose virtual velocities are zero.

1. Those whose line of action passes through an instantaneous axis; as *rolling friction*, but not sliding friction nor the resistance of any medium.

2. Those whose line of action is perpendicular to the direction of motion of the point of application; as the re-

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* For history see Nichol's *Cyclopædia*, Art. "Energy." Also *Edinburgh Philosophical Journal*, on the Science of Energetics, 1855, and *Philosophical Magazine*, 1853 and 1866.

actions of *smooth fixed surfaces*, but not those of moving surfaces.

II. Those reactions whose virtual velocities are not zero and which therefore would enter into the equation, but which disappear when joined to other reactions.

1. The reactions between particles whose distance apart remains the same; as the tensions of *inextensible strings*, but not those of elastic strings.

2. The reaction between two rigid bodies, parts of the same system, which roll on each other. It is necessary however to include both these bodies in the *same* equation of vis viva.

III. All tensions which act along inextensible strings, even though the strings are bent by passing through smooth fixed rings.

For let a string whose tension is T connect the particles m, m', and pass through a ring distant respectively r, r' from the particles. The virtual velocity is clearly

$$T\delta r + T\delta r',$$

because the tension acts along the string. But since the string is inextensible

$$\delta r + \delta r' = 0;$$

therefore the virtual velocity is zero.

If in the last case the string be extensible, and its length at any time t be ρ, we have $\rho = r + r'$, and hence the force function is $-\int T d\rho$. Let l be the unstretched length, E the coefficient of elasticity, then $T = E\frac{\rho - l}{l}$. Hence the work done by the tension when the string is stretched from the length ρ to the length ρ' is

$$\frac{E}{2l}\{(\rho - l)^2 - (\rho' - l)^2\}.$$

158. If a system be under the action of no external forces, we have $X=0$, $Y=0$, $Z=0$, and hence the vis viva of the system is constant.

If, however, the mutual reactions between the particles of the system are such as would appear in the equation of virtual moments, then the vis viva of the system will not be constant. Thus, even if the solar system were not acted on by any external forces, yet its vis viva would not be constant. For the mutual attractions between the several planets are reactions between particles whose distance does not remain the same, and hence the sum of the virtual moments will not be zero.

Again, if the earth be regarded as a body rotating about an axis and slowly contracting from loss of heat in course of time, the vis viva will not be constant, for the same reason as before. The increase of angular velocity produced by this contraction can be easily found by the conservation of areas.

159. Let gravity be the only force acting on the system. Let the axis of z be vertical, then we have $X=0$, $Y=0$, $Z=-g$. Hence the equation of vis viva becomes

$$\Sigma mv'^2 - \Sigma mv^2 = -2Mg\,(\bar{z}' - \bar{z}).$$

Thus the vis viva of the system depends only on the altitude of the centre of gravity. If any horizontal plane be drawn, the vis viva of the system is the same whenever the centre of gravity passes through the plane.

160. *To determine the vis viva of a rigid body in motion.*

If a body move in any manner its vis viva at any instant is equal to the vis viva of the whole mass collected at its centre of gravity, plus the vis viva round the centre of gravity considered as a fixed point: or

The vis viva of a body = vis viva due to translation
+ vis viva due to rotation.

Let x, y, z be the co-ordinates of a particle whose mass is m and velocity v, and let $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity G of the body. Let $x=\bar{x}+x'$, $y=\bar{y}+y'$, $z=\bar{z}+z'$. Then by a property of the centre of gravity $\Sigma mx'=0$, $\Sigma my'=0$, $\Sigma mz'=0$. Hence $\Sigma m\frac{dx'}{dt}=0$, $\Sigma m\frac{dy'}{dt}=0$, $\Sigma m\frac{dz'}{dt}=0$.

Now the vis viva of a body is

$$\Sigma mv^2=\Sigma m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}.$$

Substituting for x, y, z, this becomes

$$\Sigma m\left\{\left(\frac{d\bar{x}}{dt}\right)^2+\left(\frac{d\bar{y}}{dt}\right)^2+\left(\frac{d\bar{z}}{dt}\right)^2\right\}+\Sigma m\left\{\left(\frac{dx'}{dt}\right)^2+\left(\frac{dy'}{dt}\right)^2+\left(\frac{dz'}{dt}\right)^2\right\}$$
$$+2\frac{d\bar{x}}{dt}\Sigma m\frac{dx'}{dt}+2\frac{d\bar{y}}{dt}\Sigma m\frac{dy'}{dt}+2\frac{d\bar{z}}{dt}\Sigma m\frac{dz'}{dt}.$$

All the terms in the last line vanish as they should do by Art. 10. The first term in the first line is the vis viva of the whole mass Σm, collected at the centre of gravity. The second term is the vis viva due to rotation round the centre of gravity.

This expression for the vis viva may be put into a more convenient shape.

First. *Let the motion be in two dimensions.* Let $\bar{v}$ be the velocity of the centre of gravity, $\bar{r}$, $\bar{\theta}$ its polar co-ordinates referred to any origin in the plane of motion. Let r' be the distance of any particle whose mass is m from the centre of gravity, and let v' be its velocity relatively to the centre of gravity. Let ω be the angular velocity of the whole body about the centre of gravity, and Mk^2 its moment of inertia about the same point.

The vis viva of the whole mass collected at G is $M\bar{v}^2$, which may by the Differential Calculus be put into either of the forms

$$M\bar{v}^2=M\left\{\left(\frac{d\bar{x}}{dt}\right)^2+\left(\frac{d\bar{y}}{dt}\right)^2\right\}=M\left\{\left(\frac{d\bar{r}}{dt}\right)^2+\bar{r}^2\left(\frac{d\bar{\theta}}{dt}\right)^2\right\}.$$

The vis viva about G is $\Sigma m v'^2$. But since the body is turning about G, we have $v' = r'\omega$. Hence

$$\Sigma m v'^2 = \omega^2 . \Sigma m r'^2 = \omega^2 . M k^2.$$

Hence the whole vis viva of the body is

$$\Sigma m v^2 = M\bar{v}^2 + M k^2 \omega^2.$$

If the body be turning about an instantaneous axis, whose distance from the centre of gravity is r, we have $\bar{v} = r\omega$. Hence $$\Sigma m v^2 = M\omega^2 (r^2 + k^2) = M k'^2 \omega^2,$$
where Mk'^2 is the moment of inertia about the instantaneous axis.

Secondly. *Let the body be in motion in space of three dimensions.*

Let $\bar{v}$ be the velocity of G; $\bar{r}$, $\bar{\theta}$, $\bar{\phi}$ its polar co-ordinates referred to any origin. Let ω_x, ω_y, ω_z be the angular velocities of the body about any three axes at right angles meeting in G, and let A, B, C be the moments of inertia of the body about the axes. Let x', y', z' be the co-ordinates of a particle m referred to these axes.

The vis viva of the whole mass collected at G is $M\bar{v}^2$, which may be put into either of the following forms (see Differential Calculus):

$$M\bar{v}^2 = M\left\{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2\right\}$$

$$= M\left\{\left(\frac{d\bar{r}}{dt}\right)^2 + \bar{r}^2 \sin^2\bar{\theta}\left(\frac{d\bar{\phi}}{dt}\right)^2 + \bar{r}^2\left(\frac{d\bar{\theta}}{dt}\right)^2\right\}.$$

The vis viva due to the motion about G is

$$\Sigma m v'^2 = \Sigma m\left\{\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2\right\}.$$

But
$$\left.\begin{aligned} \frac{dx'}{dt} &= \omega_y z' - \omega_z y' \\ \frac{dy'}{dt} &= \omega_z x' - \omega_x z' \\ \frac{dz'}{dt} &= \omega_x y' - \omega_y x' \end{aligned}\right\}.$$

Substituting these values, we get, since $A = \Sigma m (y'^2 + z'^2)$, $B = \Sigma m (z'^2 + x'^2)$, $C = \Sigma m (x'^2 + y'^2)$,

$$\Sigma mv'^2 = A\omega_x^2 + B\omega_y^2 + C\omega_z^2$$
$$- 2 (\Sigma m x'y') \omega_x\omega_y - 2 (\Sigma m y'z') \omega_y\omega_z - 2 (\Sigma m z'x') \omega_z\omega_x.$$

If the axes of co-ordinates be the principal axes at G, this reduces to

$$\Sigma mv'^2 = A\omega_x^2 + B\omega_y^2 + C\omega_z^2.$$

If the body be turning about a point O, whose position is fixed for the moment, the vis viva may be proved in the same way to be

$$\Sigma mv^2 = A'\omega_x^2 + B'\omega_y^2 + C'\omega_z^2,$$

where A', B', C' are the principal moments of inertia at the point O, and ω_x, ω_y, ω_z are the angular velocities of the body about the axes.

161. Ex. 1. *A circular wire can turn freely about a vertical diameter as a fixed axis, and a bead can slide freely along it under the action of gravity. The whole system being set in rotation about the vertical axis, find the subsequent motion.*

Let M and m be the masses of the wire and bead, ω their common angular velocity about the vertical. Let a be the radius of the wire, Mk^2 its moment of inertia about the diameter. Let the centre of the wire be the origin, and let the axis of y be measured vertically downwards. Let θ be the angle the radius drawn from the centre of the wire to the bead makes with the axis of y.

It is evident, since gravity acts vertically and since all the reactions at the fixed axis must pass through the axis, that the moment of all the forces about the vertical diameter is zero. Hence, taking moments about the vertical, we have

$$Mk^2\omega + ma^2 \sin^2\theta\omega = h.$$

And by the principle of vis viva,

$$Mk^2\omega^2 + m\left\{a^2 \left(\frac{d\theta}{dt}\right)^2 + a^2 \sin^2\theta\omega^2\right\} = C + 2mga\cos\theta.$$

These two equations will suffice for the determination of $\frac{d\theta}{dt}$ and ω. Solving them, we get

$$\frac{h^2}{Mk^2 + ma^2 \sin^2 \theta} + ma^2 \left(\frac{d\theta}{dt}\right)^2 = C + 2mga \cos \theta.$$

This equation cannot be integrated, and hence θ cannot be found in terms of t. To determine the constants h and C we must recur to the initial conditions of motion. Supposing that initially $\theta = \pi$, and $\frac{d\theta}{dt} = 0$ and $\omega = \alpha$, then $h = Mk^2\alpha$ and $C = 2mga + Mk^2\alpha^2$.

162. Ex. 2. *Two equal and perfectly rough spheres are placed one on the top of the other in unstable equilibrium, the lower one resting on a perfectly smooth horizontal plane. A slight disturbance being given to the system, find the subsequent motion, supposing the centres of the spheres to move in one plane.*

Let C, C' be the centres of the lower and upper spheres, P their point of contact. Since the system is acted on by no horizontal forces, the horizontal velocity of the centre of gravity is constant (Art. 46); but the system starts from rest, hence the common centre of gravity P of the two spheres moves in a vertical straight line. Let this line be taken for the axis of y, and let the vertical plane in which the centres of the spheres move be taken for the plane of xy, and let the origin be in the fixed horizontal plane.

Let x, a be the co-ordinates of C, x', y' of C', and let θ be the acute angle CC' makes with the vertical. Let CA, $C'A'$ be those radii of the spheres which initially were in the same straight line. Then since one sphere *rolls* on the other, the angles ACP, $A'C'P$ are equal; let this angle be $= \phi$. Let F be the friction between the spheres.

Now the vis viva of the sphere $C =$ vis viva due to translation plus the vis viva due to rotation, Art. 160. The first of these is $M\left(\frac{dx}{dt}\right)^2$, and the latter is $Mk^2\left\{\frac{d(\theta - \phi)}{dt}\right\}^2$, since $\theta - \phi$ is the angle a fixed line CA in the body makes with a fixed line in space, viz. the vertical. Similarly the vis viva of the sphere C' is

$$= M\left(\frac{dx'}{dt}\right)^2 + M\left(\frac{dy'}{dt}\right)^2 + Mk^2\left\{\frac{d(\theta+\phi)}{dt}\right\}^2.$$

Now the only impressed force is gravity, therefore the force function is

$$- M\int g\, dy' = MC - Mgy'.$$

Hence, by the principle of vis viva, we have

$$\left(\frac{dx}{dt}\right)^2 + k^2\left\{\frac{d(\theta-\phi)}{dt}\right\}^2 + \left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + k^2\left\{\frac{d(\theta+\phi)}{dt}\right\}^2$$
$$= 2C - 2gy' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(1).$$

Taking moments about the centre of gravity of each sphere, we have

$$k^2\frac{d^2(\theta-\phi)}{dt^2} = \frac{Fa}{M} \;\ldots\ldots\ldots\ldots\ldots\ldots \;(2),$$

$$k^2\frac{d^2(\theta+\phi)}{dt^2} = \frac{Fa}{M} \;\ldots\ldots\ldots\ldots\ldots\ldots \;(3).$$

Also we have the geometrical equations

$$x = -a\sin\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(4),$$

$$x' = a\sin\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(5),$$

$$y' = a + 2a\cos\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(6).$$

To solve these,

Subtracting (2) from (3), $\frac{d^2\phi}{dt^2} = 0$; and $\therefore$ $\frac{d\phi}{dt} = \text{const.}$

Since the motion starts from rest we have $\frac{d\phi}{dt} = 0$. Hence ϕ is always zero, or the two spheres descend as if they were rigidly connected.

Substituting for x, x' and y' from (4), (5), (6), equation (1) becomes

$$(k^2 + a^2 + a^2\sin^2\theta)\left(\frac{d\theta}{dt}\right)^2 = ga\,(\text{const.} - \cos\theta).$$

Since the initial value of θ is $\theta = 0$, then the right-hand side of this equation is clearly $= ga(1 - \cos\theta)$. This equation gives $\frac{d\theta}{dt}$.

At the instant when the upper ball reaches the ground $\theta = \frac{\pi}{2}$, hence we then have $\left(\frac{d\theta}{dt}\right)^2 = \frac{ag}{k^2 + 2a^2}$.

163. The equation of Vis Viva may be applied to the case of relative motion in the following manner*. *Suppose the system at any instant to become fixed to the set of moving axes relative to which the motion is required, and calculate what would then be the effective forces on the system. If we apply these as additional impressed forces to the system but reversed in direction, we may use the equation of Vis Viva to determine the relative motion as if the axes were fixed in space.*

Let (x, y, z) be the co-ordinates of any particle m referred to any axes Ox, Oy, Oz fixed in space. Let $O'\xi$, $O'\eta$, $O'\zeta$ be the moving axes relative to which the motion is required. Let Δx, Δy, Δz be the variations produced in (x, y, z) by a motion of the axes $O'\xi$, $O'\eta$, $O'\zeta$, supposing the particle to be rigidly attached to them. Let dx, dy, dz be the variations produced by a motion relative to the axes $O'\xi$, $O'\eta$, $O'\zeta$, supposing them fixed for the instant. If then $d.x$, $d.y$, $d.z$ be the actual changes produced in (x, y, z) during the time dt, we have

$$d.x = dx + \Delta x,$$

with similar equations for y and z.

The general equation of motion is

$$\Sigma m\left(\frac{d^2.x}{dt^2}\delta x + \dots\right) = \Sigma m(X\delta x + \dots).$$

Let us take the arbitrary displacements δx, δy, δz to be the relative displacements dx, dy, dz. We have

$$\Sigma m\left\{\left(\frac{d^2x}{dt^2} + 2\frac{d\Delta x}{dt^2} + \frac{\Delta^2 x}{dt^2}\right)dx + \dots\right\} = \Sigma m(Xdx + \dots).$$

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* This theorem is due to Coriolis, see the *Journal Polytech.* 1831.

Let the linear motion of O' at any instant be given by the velocities (u, v, w) parallel to the fixed axes, and the angular motion of $O'\xi$, $O'\eta$, $O'\zeta$ by the angular velocities θ_1, θ_2, θ_3 about axes drawn through O' parallel to Ox, Oy, Oz. Then

$$\frac{\Delta x}{dt} = u - y\theta_3 + z\theta_2;$$

$$\therefore \frac{d\Delta x}{dt^2} = -\frac{dy}{dt}\theta_3 + \frac{dz}{dt}\theta_2;$$

because the symbol d indicates a motion relative to the moving axes, supposing them fixed for the instant. Similar equations apply for $\frac{d\Delta y}{dt^2}$ and $\frac{d\Delta z}{dt^2}$. Hence

$$\frac{d\Delta x}{dt^2}dx + \frac{d\Delta y}{dt^2}dy + \frac{d\Delta z}{dt^2}dz = 0\,^*.$$

Substituting, the equation of Vis Viva becomes

$$\Sigma m\left\{\left(\frac{d^2x}{dt^2} + \frac{\Delta^2 x}{dt^2}\right)\frac{dx}{dt} + \ldots\right\} = \Sigma m\left(X\frac{dx}{dt} + \ldots\right).$$

If Σmv^2 be the vis viva calculated for the relative motion as if the axes were fixed, we have

$$\frac{d}{dt}\Sigma mv^2 = \Sigma m\left\{\left(X - \frac{\Delta^2 x}{dt^2}\right)\frac{dx}{dt} + \ldots\right\},$$

which is the analytical expression for the theorem.

In Art. 95 a rule has been given to determine the motion of a particle constrained to rest on a curve which turns about a given fixed axis. By this theorem we may now extend the rule to the case in which the curve moves in any given manner.

164. Ex. *A sphere rolls on a perfectly rough inclined plane which turns with a uniform angular velocity* n *about a horizontal axis in its own plane. Supposing the motion of*

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[*] This result might also have been arrived at by performing Δ on the square of the relative velocity, viz.: $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$.

the sphere to take place in a vertical plane perpendicular to the axis of rotation, find the motion of the sphere relative to the plane.

Let Ox be the trace described by the sphere as it rolls on the plane, and let Oy be drawn through the axis of rotation perpendicular to Ox in the plane of motion of the sphere. Let nt be the angle Ox makes with a horizontal plane through the axis of rotation. Let ϕ be the angle that radius of the sphere which was initially perpendicular to the plane makes with the axis of y. Let (x, y) be the co-ordinates of P the centre of the sphere, and Mk^2 the moment of inertia of the sphere about a diameter.

If the sphere were fixed relatively to the plane its effective forces would be Mn^2x and Mn^2y parallel to the axes, and $Mk^2\dfrac{dn}{dt}=0$ round the centre of gravity. Also the impressed force, gravity, is equivalent to $g\sin nt$ and $-g\cos nt$ parallel to the moving axes. Hence the equation of Vis Viva for relative motion becomes

$$\frac{1}{2}\frac{d}{dt}\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+k^2\left(\frac{d\phi}{dt}\right)^2\right\}=n^2x\frac{dx}{dt}+n^2y\frac{dy}{dt}$$
$$+g\sin nt\frac{dx}{dt}-g\cos nt\frac{dy}{dt}.$$

Here $\dfrac{dx}{dt}=a\dfrac{d\phi}{dt}$ and $\dfrac{dy}{dt}=0$. We have therefore

$$\left(1+\frac{k^2}{a^2}\right)\frac{d^2x}{dt^2}=n^2x+g\sin nt.$$

This equation might have been derived from the formulæ of moving axes given in Art. 94.

This gives, if $k^2=\dfrac{2}{5}a^2$,

$$x=-\frac{5}{12}\frac{g}{n^2}\sin nt+Ae^{n\sqrt{\frac{5}{7}}t}+Be^{-n\sqrt{\frac{5}{7}}t},$$

where A, B are two constants which depend on the initial conditions of the question.

On the General Equations of Motion.

165. The principle of Virtual Velocities has been applied to obtain the equation of Vis Viva. This equation is of the first order and is a first integral of the equations of motion, free from all the unknown reactions. If the system admit of only one independent motion this equation will be sufficient for the solution of the problem. But we now propose to use the same principle in a different manner, so that even when the system admits of several independent motions we shall yet obtain equations free from all unknown reactions sufficient to determine the whole motion.

Let (x, y, z) be the co-ordinates of a particle m referred to any fixed rectangular axes, and X, Y, Z the impressed accelerating forces acting on this particle. Then we have the equation

$$\Sigma m \left\{\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \frac{d^2z}{dt^2}\,\delta z\right\} = \Sigma m\,\{X\delta x + Y\delta y + Z\delta z\},$$

where $\delta x, \delta y, \delta z$ are any small arbitrary displacements consistent with the geometrical relations at the time t, and the reactions of the system are not included among the forces on the right-hand side, see Art. 157.

In the following investigations, differential coefficients with respect to t will for the sake of brevity be denoted by accents. Thus $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ will be written x' and x''.

The quantities $x, y \ldots$ are not independent of each other, being connected by the geometrical relations of the system. But they may be expressed in terms of a certain number of independent variables whose values will determine the position of the system at any time. These have been called in Art. 41 the "co-ordinates of the system." Let these be called $\theta, \phi, \psi \ldots$. Then $x, y \ldots$ are functions of $\theta, \phi \ldots$ and t*. Let

$$x = f(t, \theta, \phi \ldots);$$

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* Lagrange only considers the case in which the geometrical relations do not contain the time explicitly, but the method has been extended to the general case by Vieille, in Liouville's *Journal*, 1849.

$$\therefore\ \delta x = A\delta\theta + B\delta\phi + \dots$$

$$x' = \alpha + A\theta' + B\phi' + \dots,$$

where α, A, $B\dots$ are written for $f'(t)$, $f'(\theta)$, $f'(\phi)\dots$. There will be similar equations for δy, y', &c.

Now $$\frac{d^2x}{dt^2}\,\delta x = \frac{d}{dt}(x'\delta x) - x'\frac{d\delta x}{dt}.$$

But $$x'\delta x = (A\alpha + A^2\theta' + AB\phi' + \dots)\,\delta\theta + \text{\&c},$$

$$x'^2 = \alpha^2 + 2\alpha A\theta' + A^2\theta'^2 + 2AB\theta'\phi' + \text{\&c.};$$

$$\therefore\ x'\delta x = \frac{1}{2}\,\frac{d(x'^2)}{d\theta'}\,\delta\theta + \text{\&c.};$$

$$\therefore\ \frac{d}{dt}(x'\delta x) = \frac{1}{2}\,\frac{d}{dt}\,\frac{d(x'^2)}{d\theta'}\,\delta\theta + \frac{1}{2}\,\frac{d(x'^2)}{d\theta'}\,\frac{d\delta\theta}{dt}$$

$$+ \text{ similar terms in } \phi,\ \psi, \text{ \&c.}$$

Again, since the operations d and δ are independent, we have $d\delta x = \delta dx$, and therefore

$$x'\frac{d\delta x}{dt} = \frac{1}{2}\,\delta(x'^2)$$

$$= \frac{1}{2}\,\frac{d(x'^2)}{d\theta}\,\delta\theta + \frac{1}{2}\,\frac{d(x'^2)}{d\theta'}\,\delta\theta'$$

$$+ \text{ similar terms in } \phi,\ \psi, \text{ \&c.}$$

Hence

$$\frac{d^2x}{dt^2}\,\delta x = \frac{1}{2}\left\{\frac{d}{dt}\,\frac{d(x'^2)}{d\theta'} - \frac{d(x'^2)}{d\theta}\right\}\delta\theta$$

$$+ \text{ similar terms in } \phi,\ \psi, \text{ \&c.}$$

It may be remarked that $\delta\theta'$ has disappeared, and it is this which makes Lagrange's transformation so successful. We have by similar reasoning

$$\frac{d^2y}{dt^2}\,\delta y = \frac{1}{2}\left\{\frac{d}{dt}\,\frac{d(y'^2)}{d\theta'} - \frac{d(y'^2)}{d\theta}\right\}\delta\theta + \text{\&c.}$$

$$\frac{d^2z}{dt^2}\,\delta z = \frac{1}{2}\left\{\frac{d}{dt}\,\frac{d(z'^2)}{d\theta'} - \frac{d(z'^2)}{d\theta}\right\}\delta\theta + \text{\&c.}$$

Let T be half the vis viva* of the body, so that

$$T = \frac{1}{2}\Sigma m\,(x'^2 + y'^2 + z'^2).$$

Then adding together the three equations above we have

$$\Sigma m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right) = \left(\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta}\right)\delta\theta$$
$$+ \text{ similar terms in } \phi, \psi, \&c.$$

Let U be the force function of the system, then

$$\Sigma m\,(X\delta x + Y\delta y + Z\delta z) = \delta U$$
$$= \frac{dU}{d\theta}\delta\theta + \text{ similar terms in } \phi, \psi, \&c.$$

By the principle of virtual velocities, these two expressions are equal. But since $\delta\theta, \delta\phi, \dots$ are independent, this equality cannot exist unless the coefficients of $\delta\theta, \delta\phi, \dots$ are separately equal. Therefore

$$\left.\begin{aligned} \frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} &= \frac{dU}{d\theta} \\ \frac{d}{dt}\frac{dT}{d\phi'} - \frac{dT}{d\phi} &= \frac{dU}{d\phi} \\ \&c. &= \&c. \end{aligned}\right\}.$$

These are called *Lagrange's general equations of motion.* They are of the second order, and their number is always equal to the number of independent quantities $\theta, \phi, \dots$ to be found. Hence they are sufficient to determine the whole motion. They are the equations we should have obtained if we had written down the ordinary equations of motion and eliminated the unknown reactions.

166. In the case of a rigid body of mass M moving in space in any manner, its semi vis viva at any time t may be

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* Coriolis, Helmholtz and others suggest that it would be more convenient if the *vis viva* were defined to be half the sum of the products of the masses into the squares of the velocities. See *Phil. Trans.* 1854, p. 89.

expressed in terms of the co-ordinates of its centre of gravity, and the angles θ, ϕ, ψ, which the principal axes at the centre of gravity make with some fixed axes in the manner explained in Art. 109. In this case we have by Arts. 109, and 160.

$$2T = M(x'^2 + y'^2 + z'^2) + C(\phi' + \psi' \cos\theta)^2$$
$$+ (A \sin^2\phi + B\cos^2\phi)\,\theta'^2 + \sin^2\theta\,(A\cos^2\phi + B\sin^2\phi)\,\psi'^2$$
$$+ 2(B - A)\sin\theta\sin\phi\cos\phi\,\theta'\psi'.$$

In the case in which two of the principal moments A and B are equal, this takes the simpler form

$$2T = M(x'^2 + y'^2 + z'^2) + C(\phi' + \psi'\cos\theta)^2$$
$$+ A\theta'^2 + A\sin^2\theta\psi'^2.$$

The independent variables in terms of which the motion is to be found should be chosen so as to simplify the expression for the vis viva as much as possible. They must however be such that the co-ordinates of every particle of the body could, if necessary, be expressed in terms of them by means of equations which do not contain any differential coefficients with regard to t.

167. As an example of the use of Lagrange's equations, let us apply them to the following problem.

A body, two of whose principal moments at the centre of gravity are equal, turns about a fixed point O *situated in the axis of unequal moment under the action of gravity. To determine the conditions that there may be a simple equivalent pendulum.*

Def. If a body be suspended from a fixed point O under the action of gravity, and if the angular motion of the straight line joining O to the centre of gravity be the same as that of a string of length l to the extremity of which a heavy particle is attached, then l is called the *length of the simple equivalent pendulum.* This is an extension of the definition in Art. 54.

Let OC be the axis of unequal moment, A, B, C the principal moments at the fixed point, and let the rest of the notation be the same as in Art. 166. Then

$$2T = A(\theta'^2 + \sin^2\theta\psi'^2) + C(\phi' + \psi'\cos\theta)^2,$$

$$U = Mgh\cos\theta + \text{constant},$$

where h is the distance of the centre of gravity from the fixed point, and gravity is supposed to act in the positive direction of the axis of z. Lagrange's equations will be found to become

$$\frac{d}{dt}(A\theta') - A\sin\theta\cos\theta\psi'^2 + C\psi'(\phi' + \psi'\cos\theta)\sin\theta = -Mgh\sin\theta,$$

$$\frac{d}{dt}\{C(\phi' + \psi'\cos\theta)\} = 0,$$

$$\frac{d}{dt}\{C(\phi' + \psi'\cos\theta)\cos\theta + A\sin^2\theta\psi'\} = 0.$$

Integrating the last two equations we have

$$\phi' + \psi'\cos\theta = n,$$

where n is some constant expressing the angular velocity about the axis of unequal moment; and

$$Cn\cos\theta + A\sin^2\theta\frac{d\psi}{dt} = \alpha,$$

where α is another constant expressing the moment of the momentum about the vertical through O.

Instead of the first equation, we may use the equation of vis viva, which gives

$$A\left\{\sin^2\theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2\right\} = \beta + 2Mgh\cos\theta.$$

To determine the arbitrary constants α and β we must have recourse to the initial values of θ and ψ. Let θ_0, ψ_0, $\frac{d\theta_0}{dt}$, $\frac{d\psi_0}{dt}$

be the initial values of θ, ψ, $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$, then the above equations become

$$\left.\begin{aligned} &\sin^2\theta\frac{d\psi}{dt}+\frac{Cn}{A}\cos\theta=\sin^2\theta_0\frac{d\psi_0}{dt}+\frac{Cn}{A}\cos\theta_0 \\ &\sin^2\theta\left(\frac{d\psi}{dt}\right)^2+\left(\frac{d\theta}{dt}\right)^2=\sin^2\theta_0\left(\frac{d\psi_0}{dt}\right)^2+\left(\frac{d\theta_0}{dt}\right)^2+2\frac{Mgh}{A}(\cos\theta-\cos\theta_0) \end{aligned}\right\}(1).$$

These equations, when solved, give θ and ψ in terms of t, and thus determine the motion of the line OG. The corresponding equations for the motion of the simple equivalent pendulum OL are found by making $C=0$, $A=Ml^2$, and $h=l$, where l is the length of the pendulum. This gives

$$\left.\begin{aligned} &\sin^2\theta\frac{d\psi}{dt}=\sin^2\theta_0\frac{d\psi_0}{dt} \\ &\sin^2\theta\left(\frac{d\psi}{dt}\right)^2+\left(\frac{d\theta}{dt}\right)^2=\sin^2\theta_0\left(\frac{d\psi_0}{dt}\right)^2+\left(\frac{d\theta_0}{dt}\right)^2+2\frac{g}{l}(\cos\theta-\cos\theta_0) \end{aligned}\right\}(2).$$

In order that the motions of the two lines OG and OL may be the same, the two equations (1) and (2) must be the same. This will be the case if either $Cn=0$, or $\theta=\theta_0$. In the first case, we must have $n=0$, or $C=0$, so that the body must either have no rotation about OG, or else the body must be a rod. In the second case, we must have throughout the motion θ and $\frac{d\psi}{dt}$ constant, so that the body must be moving in steady motion making a constant angle with the vertical. See Art. 240. In either case, the two sets of equations are identical if $l=\frac{A}{Mh}$. This is the same formula which was obtained in Art. 54.

168. In some cases it may be found more convenient to express the motion of the system in terms of variables $\theta, \phi \ldots$ which are not all independent. These variables will then be connected by certain geometrical relations. Let these be

$$\left.\begin{aligned} L(\theta, \phi, \ldots\ldots t)&=0 \\ M(\theta, \phi, \ldots\ldots t)&=0 \\ \&c.&=0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(1),$$

which are supposed not to contain θ', ϕ' .... See Art. 166. The general equation of virtual velocities may be transformed by the same reasoning as before into the equation

$$\Sigma\left[\frac{d}{dt}\,\frac{dT}{d\theta'}-\frac{dT}{d\theta}-\frac{dU}{d\theta}\right]\delta\theta=0\;.........\;(2),$$

where the symbol Σ means summation for all the co-ordinates θ, ϕ, &c. Also we have

$$\left.\begin{aligned}\Sigma\frac{dL}{d\theta}\delta\theta&=0\\ \&c.&=0\end{aligned}\right\}\;....................\;(3).$$

Let us use the method of indeterminate multipliers. The equations will then be transformed by the usual process into *

$$\frac{d}{dt}\,\frac{dT}{d\theta'}-\frac{dT}{d\theta}=\frac{dU}{d\theta}+\lambda\frac{dL}{d\theta}+\mu\frac{dM}{d\theta}+\;............\;(4),$$

with similar equations for ϕ and ψ. Besides these we have the geometrical equations (1); these are together sufficient to determine the symbols θ, ϕ, &c. λ, μ, &c.

It follows, that if the position of the system has not been expressed in terms of the fewest possible number of symbols, yet in forming the dynamical equations according to Lagrange's rule, we may treat these as if they were independent, provided we add to dU, the differential of the force function, the quantities λdL, μdM, &c.

The method of indeterminate multipliers is really a reintroduction of the unknown reactions into the dynamical equations. Thus suppose the geometrical relation $L=0$ to

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* If we multiply each of the geometrical equations (3) by λ, μ, ..., respectively and subtract them from (2), we get

$$\Sigma\left[\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}-\frac{dU}{d\theta}-\lambda\frac{dL}{d\theta}-\mu\frac{dM}{d\theta}-\dots\right]\delta\theta=0.$$

Now there will be as many indeterminate multipliers λ, μ... as there are geometrical equations, i.e. as there are dependent variables among the co-ordinates θ, ϕ... By properly choosing λ, μ... the coefficients of the differentials of these variables may be made to vanish, and then the coefficients of the others will vanish of themselves.

express the fact that the particle m is constrained to rest on the surface $L(x, y, z) = 0$. Then $\lambda \frac{dL}{dx}$, $\lambda \frac{dL}{dy}$, $\lambda \frac{dL}{dz}$ are the resolved parts of the reaction of this surface parallel to the axes. At the same time, the process introduces these reactions in just that manner which is most convenient for the solution of the problem.

169. The transformation on which Lagrange's equations are founded may be generalized in a manner which puts the whole method in a clearer light. Let T be any function of $x, x', x'' \ldots y, y', y'' \ldots\ldots$ &c., and of t, and let it be required to transform the variables x, y, &c. into others θ, ϕ, &c. by writing for x, y, &c. any functions of θ, ϕ, &c. and t. For the sake of brevity let the partial differential coefficients $\frac{dT}{dx}, \frac{dT}{dx'}, \frac{dT}{dx''}$, &c. be denoted by X, X_1, X_2, &c. then similarly $\frac{dT}{dy}, \frac{dT}{dy'}$ will be represented by Y, Y_1, and as before let accents denote total differential coefficients with regard to t. Let δ be a symbol of differentiation which does not operate on t, so that δT means the variation produced in T by any arbitrary alteration of x, y, &c., t remaining unaltered.

The function T may be expressed in two ways, either as a function of x, y, &c. and t, or of θ, ϕ, &c. and t. By comparing the two values of $\delta \int_{t_0}^{t_1} T dt$ given by the Calculus of Variations for these two methods of expressing T we get the equation*,

$$\left.\begin{array}{c}(X - X_1' + X_2'' - \text{\&c.})\,\delta x \\ + \text{similar expressions} \\ \text{for } y, z, \text{\&c.}\end{array}\right\} = \left\{\begin{array}{c}(\Theta - \Theta_1' + \Theta_2'' - \text{\&c.})\,\delta\theta \\ + \text{similar expressions} \\ \text{for } \phi, \psi, \text{\&c.}\end{array}\right.$$

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* The two equal values of $\delta \int T dt$ are

$$\Sigma\left[\int (X - X_1' + \ldots)\,\delta x\, dt + (X_1 - X_2' + \ldots)\,\delta x + \text{\&c.}\right]$$
$$= \Sigma\left[\int (\Theta - \Theta_1' + \ldots)\,\delta\theta\, dt + (\Theta_1 - \Theta_2' + \ldots)\,\delta\theta + \text{\&c.}\right],$$

where the symbol Σ means summation of similar terms for all the variables.

In the particular case in which T is half the vis viva we have

$$T = \frac{1}{2} \Sigma m \left(x'^2 + y'^2 + z'^2\right),$$

therefore $X = 0$, $X_1 = x'$, $X_2 = 0$, &c. and the equation becomes

$$-\Sigma m \left(x'' \delta x + \text{\&c.}\right) = \left(\frac{dT}{d\theta} - \frac{d}{dt}\frac{dT}{d\theta'}\right) d\theta + \text{\&c.}$$

170. In order to conform to modern notation, let us represent the independent co-ordinates of the dynamical system by q_1, $q_2 \ldots q_n$ and their differential coefficients by q_1', $q_2' \ldots q_n'$. Let T be half the vis viva and U the force function as before. Also let $L = T + U$, then Lagrange's equations may be written in the form

$$\frac{d}{dt}\frac{dL}{dq'} - \frac{dL}{dq} = 0,$$

where q stands for any one of the co-ordinates q_1, $q_2 \ldots q_n$. These equations have been put into another form by Sir W. Hamilton, which is found to be more convenient for the investigation of the general properties of any dynamical system. Let us put $\frac{dL}{dq_1'} = p_1$, $\frac{dL}{dq_2'} = p_2$, &c., or, as we may more

Now the variations $\delta\theta$, $\delta\phi$, &c. are all arbitrary, let them be chosen so that they vanish at both limits. Then since

$$\delta x = \frac{dx}{d\theta}\delta\theta + \frac{dx}{d\phi}\delta\phi + \ldots$$

δx must also vanish at both limits, and the same reasoning will apply to all the other variables. Hence we have

$$\Sigma \int (X - X_1' + \ldots)\, \delta x\, dt = \Sigma \int (\Theta - \Theta_1' + \ldots)\, \delta\theta\, dt.$$

Now substitute for δx, δy, &c. their values in terms of $\delta\theta$, $\delta\phi$, &c. Then by the usual reasoning in the Calculus of Variations, since $\delta\theta$, $\delta\phi$, &c. are independent and arbitrary, their coefficients under the integral signs must be equal each to each. But these are just the conditions that

$$\Sigma (X - X_1' + \ldots)\, \delta x = \Sigma (\Theta - \Theta_1' + \ldots)\, \delta\theta.$$

We might also reason thus—By comparing the two values of $\delta\int T dt$ we see that the integral of $\Sigma[(X - \text{\&c.})\,\delta x - (\Theta - \text{\&c.})\,\delta\theta]$ may be completely found. Hence this expression must be a perfect differential with regard to t, quite independently of the operation δ. But this cannot be, unless it vanishes, because it contains only the variations δx, $\delta\theta$, &c. and not the differential coefficients of these variations.

briefly write it, $\frac{dL}{dq'}=p$. The quantity L is a function of the letters represented by q and q'. The object of the Hamiltonian transformation is to change the independent variables to the letters represented by p and q. To effect this, let

$$H=-L+\Sigma pq',$$

where the symbol Σ implies a summation of similar terms from $p=p_1$ to $p=p_n$. We shall suppose that by help of the n equations represented by $\frac{dL}{dq'}=p$, H has been expressed as a function of the p's and q's only. Taking* the total differential of H we have

$$dH=\Sigma\left[-\frac{dL}{dq}dq-\frac{dL}{dq'}dq'+pdq'+q'dp\right].$$

But by hypothesis $\frac{dL}{dq'}=p$, and by Lagrange's equations $\frac{dL}{dq}=p'$; hence by substitution

$$dH=\Sigma\{q'dp-p'dq\}.$$

This equation implies the partial differential equations

$$\frac{dH}{dp}=q', \qquad \frac{dH}{dq}=-p'.$$

These are called the Hamiltonian equations.

If we refer to the first demonstration of Lagrange's equations in Art. 165, we shall see that when the geometrical equations do not contain the time explicitly, T is a homogeneous function of $q_1', q_2' \dots q_n'$ of the second order, so that

$$2T=\Sigma q'\frac{dT}{dq'}$$
$$=\Sigma q'p.$$

In this case therefore we have $H=T-U$.

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* This mode of conducting the transformation is due to Prof. Donkin. It has the advantage of not assuming that the geometrical equations do not contain the time explicitly.

171. If we could solve generally the Lagrangian equations, we should determine the motion of any dynamical system whatever. This, however, we cannot do. The first step was taken by Hamilton, who showed that, when the geometrical equations do not contain the time explicitly, the complete solution may be made to depend on the determination of a single function V which he called "the characteristic function."

Let T be half the vis viva at the time t, U the force function. Then by the principle of vis viva we have

$$T = U + h,$$

where h is some constant depending on the initial conditions. Let $q_1, q_2 \dots q_n$ be the co-ordinates of the system, and let the initial values of q, q' be represented by a, a'. Let us now vary all the initial conditions, that is, let us suppose the system to be "started" in any manner slightly differing from its actual motion at the time $t = 0$. Then at the time t all the quantities represented by q, q' will have values slightly differing from the values they have with the given initial conditions. Let these differences be represented by δq, $\delta q'$, thus the symbol δ, for the moment, represents an arbitrary differentiation in which t is supposed constant. Then we have

$$\delta \int_0^t T dt = \delta \int_0^t U dt + t\delta h.$$

Expressing these variations at length according to the rules of the calculus of variations* we have

$$\Sigma \left[\frac{dT}{dq'}\delta q\right]_0^t + \Sigma \int_0^t \left(\frac{dT}{dq} - \frac{d}{dt}\frac{dT}{dq'}\right)\delta q \, dt$$

$$= \Sigma \int^t \frac{dU}{dq}\delta q \, dt + t\delta h,$$

where, as before, the symbol Σ means summation for all letters from $q = q_1$ to $q = q_n$.

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* The proof is as follows. We have

$$\delta T = \Sigma \frac{dT}{dq}\delta q + \Sigma \frac{dT}{dq'}\delta q'.$$

By Lagrange's equations this becomes

$$\Sigma\left[\frac{dT}{dq'}\delta q\right]_0^t = 2\Sigma\int_0^t \frac{dU}{dq}\delta q\,dt + t\delta h,$$

$$= 2\delta\int_0^t T dt - t\delta h.$$

Now let $V = 2\int_0^t T dt$, so that V is the whole accumulation of vis viva, *i.e.* the "action" of the system in passing from any position called its initial position to its position at the time t. Then the above equation written at length becomes

$$\delta V = \Sigma\frac{dT}{dq'}\delta q - \Sigma\frac{dT_0}{da'}\delta a + t\delta h,$$

where T_0 is the same function of a, a' that T is of q, q'.

Let us now give an extension of meaning to the symbol δ. Let δV be the difference between the "action" along two neighbouring courses which are not described in the same time. Let these times be t and $t + \delta t$. Then we must add to the expression which has just been found for δV the quantity $2T\delta t$ which is the action described by the system as it

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Now $$\int\frac{dT}{dq'}\delta q'\,dt = \int\frac{dT}{dq'}d\delta q$$

$$= \frac{dT}{dq'}\delta q - \int\frac{d}{dt}\left(\frac{dT}{dq'}\right)\delta q\,dt,$$

by integrating by parts;

$$\therefore\ \delta\int_0^t T dt = \int_0^t \delta T\,dt = \left[\Sigma\frac{dT}{dq'}\delta q\right]_0^t + \int_0^t \Sigma\left(\frac{dT}{dq} - \frac{d}{dt}\frac{dT}{dq'}\right)\delta q\,dt.$$

The investigation in the text might also be arranged so as not to require Lagrange's equations. If (x, y, z) be the Cartesian co-ordinates of a particle m, we have $2T = \Sigma m(x'^2 + y'^2 + z'^2)$ and $\Sigma m(x''\delta x + \&c.) = \Sigma m\left(\frac{dU}{dx}\delta x + \&c.\right)$. Then proceeding, step for step, as in the text, we get

$$\delta V = \Sigma m(x'\delta x + \&c.) - \Sigma m(x'_0\delta x_0 + \&c.) + t\delta h,$$

where x_0, x_0', &c. are the initial values of x, x', &c. Referring to the note in page 244, we have, equating the parts between limits,

$$\Sigma[(X_1 - X'_2 + \ldots)\delta x + \&c.] = \Sigma[(\Theta_1 - \Theta'_2 + \ldots)\delta\theta + \&c.].$$

By this theorem the expression for δV may be changed into the final form given in the text. This theorem is however only another form of Lagrange's equations.

moves along its course during the additional time δt. Again, according to our extended meaning of the symbol δ, we must represent by $q + \delta q$ the terminal co-ordinates, *i.e.* the co-ordinates in the displaced motion at the time $t + \delta t$. Hence the co-ordinates at the time t will be represented by $q + \delta q - q'\delta t$. The value of δV is therefore

$$\delta V = \Sigma \frac{dT}{dq'}(\delta q - q'\delta t) - \Sigma \frac{dT_0}{da'}\delta a + t\delta h + 2T\delta t.$$

But since T is a homogeneous function of q_1', q_2', &c. we have $2T = \Sigma \frac{dT}{dq'} q'$,

$$\therefore\ \delta V = \Sigma \frac{dT}{dq'}\delta q - \Sigma \frac{dT_0}{da'}\delta a + t\delta h.$$

If we supposed the two courses not to begin as well as not to terminate at the same instant, we may, in the same way, prove this formula to be still true.

It is clear that we may regard V as a function of t, the initial co-ordinates $a_1, a_2 \ldots a_n$ and the initial velocities $a_1', a_2' \ldots a_n'$. Also $q_1 \ldots q_n$ are functions of t and the same initial conditions. Hence by eliminating $a_1' \ldots a_n'$ we may regard V as a function of t and $a_1 \ldots a_n$, $q_1 \ldots q_n$ the co-ordinates of the system at the times $t = 0$ and $t = t$. If we use the initial equation of vis viva we may eliminate t also and consider V to be a function of h and of the co-ordinates of the system at the beginning and end of the motion considered.

Taking this latter view, we have

$$\delta V = \Sigma \frac{dV}{dq}\delta q + \Sigma \frac{dV}{da}\delta a + \frac{dV}{dh}\delta h,$$

by the Differential Calculus. Hence by comparing the two expressions for δV we find

$$\left.\begin{aligned} \frac{dV}{dq} &= \frac{dT}{dq'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}, \\ \frac{dV}{da} &= -\frac{dT_0}{da'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}, \\ \frac{dV}{dh} &= t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(C)}. \end{aligned}\right\}$$

In these equations we must write for q in the denominator successively $q_1, q_2 \dots q_n$. We thus obtain two sets of equations, corresponding to (A) and (B), the first and second of these, and the last equation remains unaltered. Suppose now V were known, then the first set of these equations when h has been eliminated by means of the last equation gives a set of first integrals of the equations of motion containing only the first differential coefficients of the co-ordinates of the bodies forming the system. The second set of equations treated similarly gives a set of second integrals which completely determine the values of $q_1, q_2 \dots q_n$ in terms of t, and their initial values and velocities.

172. There are two differential equations which the function V must satisfy. These may be found in the following manner. Let the expression for half the vis viva be

$$T = \tfrac{1}{2} A_{11} q_1'^2 + A_{12} q_1' q_2' + \dots$$

where the suffix of A in any term is made up of the suffixes of the other letters which occur in the term of which it is the coefficient. Then A_{11}, A_{12}, &c. are functions of $q_1, q_2 \dots q_n$ only. The equations (A) then become

$$\left.\begin{aligned} \frac{dV}{dq_1} &= A_{11} q_1' + A_{12} q_2' + \dots\dots\dots \\ \frac{dV}{dq_2} &= A_{21} q_1' + A_{22} q_2' + \dots\dots\dots \\ \&\text{c.} &= \&\text{c.} \end{aligned}\right\} \dots\dots\dots\dots \text{(A}'\text{)}.$$

Also since T is a homogeneous function of the second order in $q_1', q_2' \dots q_n'$ we have

$$2T = \frac{dT}{dq_1'} q_1' + \frac{dT}{dq_2'} q_2' + \dots$$

Hence the equation of vis viva may be written in the form

$$2(U + h) = \frac{dV}{dq_1} q_1' + \frac{dV}{dq_2} q_2' + \dots\dots\dots\dots \text{(D)}.$$

The quantities $q_1', q_2' \dots q_n'$ may be eliminated from the equations (A′) and (D), and we have a partial differential equation, which may be written in the form of a determinant, to find V.

Also if we consider the initial value of T, we shall have another equation of a similar form with $a_1, a_2 \dots a_n$ written instead of $q_1, q_2 \dots q_n$. It is necessary that the function V should satisfy both these equations.

If we wrote $p_1, p_2 \dots$ for $\frac{dV}{dq_1}, \frac{dV}{dq_2} \dots$ this process of finding $q_1' q_2' \dots$ from the equations (A′) and substituting in the equation of vis viva exactly corresponds to the Hamiltonian process of expressing $H = T - U$ in terms of $p_1, p_2 \dots q_1, q_2 \dots$ If then we write the Hamiltonian function H in the form

$$H(q_1 q_2 \dots q_n p_1 p_2 \dots p_n),$$

the differential equation to find V will be

$$H\left(q_1 q_2 \dots q_n \frac{dV}{dq_1} \frac{dV}{dq_2} \dots \frac{dV}{dq_n}\right) = h.$$

173. By the preceding reasoning all the integrals of a dynamical system of equations can be expressed in terms of the differential coefficients of a single function V. But the method furnishes no means of discovering this function *à priori*. It is indeed shown that it is a similar function of the initial and final co-ordinates of the system, and that it must satisfy a certain partial differential equation, but there is no rule given to determine which integral expresses the whole action. This rule has been supplied by Jacobi in the following proposition.

Let V be defined to be any function of the co-ordinates $q_1, q_2 \dots q_n$ of the system which satisfies the differential equation

$$H\left(q_1, q_2 \dots q_n \frac{dV}{dq_1} \frac{dV}{dq_2} \dots \frac{dV}{dq_n}\right) = h.$$

Suppose a solution to have been found containing $n-1$

constants* besides h, and the constant which may be introduced by simple addition to the function V. These need not be the initial values of $q_1, q_2 \dots q_n$, but may be any constants whatever. Let them be denoted by $\alpha_1, \alpha_2 \dots \alpha_{n-1}$, so that

$$V = f(q_1, q_2 \dots q_n, \quad \alpha_1, \alpha_2 \dots \alpha_{n-1}) + \alpha_n.$$

Then the integrals of the dynamical equations will be

$$\frac{df}{d\alpha_1} = \beta_1, \text{ \&c.}, \frac{df}{d\alpha_{n-1}} = \beta_{n-1} \dots\dots\dots\dots\dots (\text{B}''),$$

$$\frac{df}{dh} = t + \epsilon \dots\dots\dots\dots\dots\dots\dots\dots\dots (\text{C}''),$$

where $\beta_1, \beta_2 \dots \beta_{n-1}$ and ϵ are n new arbitrary constants. And the first integrals of the equations may be written in the form

$$\frac{df}{dq_1} = \frac{dT}{dq_1'}, \quad \frac{df}{dq_2} = \frac{dT}{dq_2'}, \text{ \&c.} = \text{\&c.} \quad \dots\dots\dots (\text{A}'').$$

Let $Q_1, Q_2 \dots Q_n$ be such functions of $q_1, q_2 \dots q_n$ and the constants, that they may satisfy *identically* the n equations

$$\left.\begin{aligned} \frac{df}{dq_1} &= A_{11} Q_1 + A_{12} Q_2 + \dots \\ \frac{df}{dq_2} &= A_{21} Q_1 + A_{22} Q_2 + \dots \\ \text{\&c.} &= \text{\&c.} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (\text{A}''').$$

Then from the mode in which the differential equation to find V has been formed, in Art. 172, we know Q_1, Q_2 will also satisfy *identically* the equation

$$U + h = \frac{1}{2} A_{11} Q_1^2 + A_{12} Q_1 Q_2 + \dots\dots\dots\dots\dots (1).$$

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* An integral of a partial differential equation has been called by Lagrange "complete," when it contains as many arbitrary constants as there are independent variables. It is implied that the constants enter in such a manner into the integral that they cannot by any algebraic process be reduced to a smaller number. For instance, if two of the constants enter in the form $\alpha_1 + \alpha_2$, they amount on the whole to only one.

First, we shall prove that $Q_1 = q_1'$, $Q_2 = q_2'$, &c., it will then follow that the equations (A″) are satisfied. Differentiating equations (B″) and (C″), we have

$$\left.\begin{aligned} \frac{d^2 f}{d\alpha_1\, dq_1}\frac{dq_1}{dt} + \frac{d^2 f}{d\alpha_1\, dq_2}\frac{dq_2}{dt} + \ldots &= 0 \\ \&\text{c.} \qquad &= 0 \\ \frac{d^2 f}{dh\, dq_1}\frac{dq_1}{dt} + \frac{d^2 f}{dh\, dq_2}\frac{dq_2}{dt} + \ldots &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (2).$$

These are the equations to find $\frac{dq_1}{dt}$, $\frac{dq_2}{dt}$, &c.

But differentiating (A‴) with regard to α_1, we have

$$\left.\begin{aligned} \frac{d^2 f}{d\alpha_1\, dq_1} &= A_{11}\frac{dQ_1}{d\alpha_1} + A_{12}\frac{dQ_2}{d\alpha_1} + \ldots \\ \frac{d^2 f}{d\alpha_1\, dq_2} &= A_{21}\frac{dQ_1}{d\alpha_1} + A_{22}\frac{dQ_2}{d\alpha_1} + \ldots \\ \&\text{c.} &= \&\text{c.} \end{aligned}\right\} \ldots\ldots\ldots\ldots (3),$$

because A_{11}, A_{12}, &c. are not functions of the constants. Multiplying these equations by Q_1, $Q_2 \ldots$, and adding, we get

$$\frac{d^2 f}{d\alpha_1\, dq_1} Q_1 + \frac{d^2 f}{d\alpha_1\, dq_2} Q_2 + \ldots = \frac{d}{d\alpha_1}\left\{\frac{1}{2} A_{11} Q_1^2 + A_{12} Q_1 Q_2 + \ldots\right\}.$$

Since the equation (1) is an identical equation the quantity in brackets on the right-hand side does not contain α_1, being equal to $U + h$. Hence the expression on the left-hand side vanishes. Thus we have an equation connecting Q_1, $Q_2 \ldots$ exactly similar to the first of equations (2). Similarly by differentiating equations (A‴) with respect to $\alpha_2 \ldots h$ successively, we shall have equations similar to the second, &c. and last of equations (2). We have therefore exactly the same equations to find Q_1, $Q_2 \ldots$ and q_1', $q_2' \ldots$. Hence $Q_1 = q_1'$, $Q_2 = q_2'$, &c.

Secondly, we shall prove that (B″) and (C″) satisfy the equations of motion. Let us consider the equation of Lagrange *,

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* We may also show that the Jacobian integrals satisfy the Hamiltonian form of the equations of motion. The peculiar relation of the differential equa-

$$\frac{d}{dt}\frac{dT}{dq_1'}-\frac{dT}{dq_1}=\frac{dU}{dq_1}.$$

When $q_1', q_2' \ldots q_n'$ have been expressed in terms of $q_1, q_2 \ldots q_n$ and the constants by means of equations (A″), we have *identically*

$$U+h=\frac{1}{2}A_{11}q_1'^2+A_{12}q_1'q_2'+\ldots$$

Therefore, differentiating partially,

$$\begin{aligned}\frac{dU}{dq_1}&=\frac{1}{2}\frac{dA_{11}}{dq_1}q_1'^2+\frac{dA_{12}}{dq_1}q_1'q_2'+\ldots\\ &+\left\{A_{11}\frac{dq_1'}{dq_1}+A_{12}\frac{dq_2'}{dq_1}+\ldots\right\}q_1'\\ &+\left\{A_{12}\frac{dq_1'}{dq_1}+A_{22}\frac{dq_2'}{dq_1}+\ldots\right\}q_2'\\ &+\&c.\end{aligned}$$

But differentiating (A″) written at length, with regard to q_1, we have

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tion to the Hamiltonian function H adapts it to this process. If we substitute the value of V given at the top of page 252 in the differential equation in page 251, the result is an identical equation. Differentiating this identity with regard to each of the n constants and replacing $\frac{dV}{dq}$ by p, we get n equations of the form

$$\frac{dH}{dp_1}\frac{d^2f}{dq_1\,d\alpha}+\frac{dH}{dp_2}\frac{d^2f}{dq_2\,d\alpha}+\ldots=0$$

to find $\frac{dH}{dp_1}$, $\frac{dH}{dp_2}$, &c. These are the same as the equations (2) in the text, hence $\frac{dH}{dp}=q'$. Again, differentiating H partially with regard to q_1, we have

$$\frac{dH}{dq_1}+\frac{dH}{dp}\frac{d^2f}{dq_1^2}+\frac{dH}{dp_2}\frac{d^2f}{dq_1\,dq_2}+\ldots=0.$$

But all the terms of this equation except the first are together equal to the total differential coefficient $\frac{dp_1}{dt}$. Hence $\frac{dH}{dq_1}=-\frac{dp_1}{dt}$. The investigations of Hamilton and Jacobi apply to a system of free particles mutually attracting each other referred to Cartesian co-ordinates. In the text the reasoning has been applied to a system of bodies referred to any co-ordinates.

$$A_{11}\frac{dq_1'}{dq_1}+A_{12}\frac{dq_2'}{dq_1}+\ldots=\frac{d^2f}{dq_1^2}-q_1'\frac{dA_{11}}{dq_1}-q_2'\frac{dA_{12}}{dq_1}-\ldots$$

$$A_{21}\frac{dq_1'}{dq_1}+A_{22}\frac{dq_2'}{dq_1}+\ldots=\frac{d^2f}{dq_1\,dq_2}-q_1'\frac{dA_{21}}{dq_1}-q_2'\frac{dA_{22}}{dq_1}-\ldots$$

$$\&\text{c.}=\&\text{c.}$$

Hence, substituting

$$\frac{dU}{dq_1}=\frac{d^2f}{dq_1^2}q_1'+\frac{d^2f}{dq_1\,dq_2}q_2'+\ldots$$
$$-\frac{1}{2}\frac{dA_{11}}{dq_1}q_1'^2-\frac{dA_{12}}{dq_1}q_1'q_2'-\ldots.$$

Next let us consider the expression for T, we see that the partial differential coefficient

$$\frac{dT}{dq_1}=\frac{1}{2}\frac{dA_{11}}{dq_1}q_1'^2+\frac{dA_{12}}{dq_1}q_1'q_2'+\ldots$$

is the same as the second line of the expression for $\frac{dU}{dq_1}$.

Also $\frac{dT}{dq_1'}=\frac{df}{dq_1}$, therefore taking the total differential coefficient, we have

$$\frac{d}{dt}\frac{dT}{dq_1'}=\frac{d^2f}{dq_1^2}q_1'+\frac{d^2f}{dq_1\,dq_2}q_2'+\ldots,$$

which is the same as the first line of the expression for $\frac{dU}{dq_1}$. Hence the differential equation of motion is satisfied.

We have also, since T is homogeneous,

$$2T=\frac{dT}{dq_1'}q_1'+\frac{dT}{dq_2'}q_2'+\ldots$$
$$=\frac{df}{dq_1}q_1'+\frac{df}{dq_2}q_2'+\ldots$$
$$=\frac{df}{dt},$$

where the differential coefficient is total. This shows that the function f represents the whole accumulated "action" in the time t. See Art. 171.

174. Jacobi has extended his theorem to the case in which the geometrical equations do contain the time explicitly. But for this we have no space. It is no part of the plan of this book to enter on Theoretical Dynamics. We cannot therefore do more than allude to Professor Donkin's theorem that a knowledge of half the integrals of the Hamiltonian system will in certain cases lead to a determination of the rest.

In Boole's *Differential Equations* it is shown that when the Hamiltonian equations are four in number, and one integral besides Vis Viva is known, both the remaining integrals can be found by integrating an exact differential equation. Miscellaneous Exercises, No. 15.

Variation of the Elements.

175. Let the integrals of a dynamical problem be

$$\left.\begin{array}{l} c_1 = f_1(p_1, q_1, p_2, q_2, \dots t) \\ c_2 = f_2(p_1, q_1, p_2, q_2, \dots t) \\ \&\text{c.} = \&\text{c.} \end{array}\right\} \dots\dots\dots\dots\dots\dots (1),$$

where $p, q, \dots$ are some variables which determine the position and motion of the system, and which are such that the equations of motion may be written in the forms

$$p' = -\frac{dH}{dq}, \quad q' = \frac{dH}{dp} \dots\dots\dots\dots\dots\dots (2),$$

in the manner explained in Art. 170. Let the equations of motion of a second dynamical problem be

$$p' = -\frac{dH}{dq} - \frac{dK}{dq}, \quad q' = \frac{dH}{dp} + \frac{dK}{dp} \dots\dots\dots (3),$$

where K is some function of $p, q \dots t$. If we consider

c_1, c_2, ... the constants of the solution of the first problem to be functions of p, q, and t, we may suppose the solution of the second problem to be represented by integrals of the same form (1) as those of the first problem. It is therefore our object to discover what functions c_1, c_2, ... are of p, q, and t. The function K is called "the disturbing function," and is usually small as compared with H.

Since the equations (1) are the integrals of the differential equations (2), we shall obtain identical expressions by substituting from (1) in (2). Hence differentiating (1), and substituting for p' and q' their values given by (2), we get

$$\left.\begin{aligned} 0 &= -\frac{dc_1}{dp}\frac{dH}{dq} + \frac{dc_1}{dq}\frac{dH}{dp} + \ldots + \frac{dc_1}{dt} \\ 0 &= \&c. \end{aligned}\right\} \qquad \text{(4)}.$$

But when c_1, c_2, ... are considered as variables, the equations (1) are the integrals of the differential equations (3). Hence repeating the same process, we have

$$\left.\begin{aligned} \frac{dc_1}{dt} &= -\frac{dc_1}{dp}\frac{dH}{dq} + \frac{dc_1}{dq}\frac{dH}{dp} + \ldots + \frac{dc_1}{dt} \\ &\quad -\frac{dc_1}{dp}\frac{dK}{dq} + \frac{dc_1}{dq}\frac{dK}{dp} + \ldots\ldots \\ \frac{dc_2}{dt} &= \&c. \end{aligned}\right\},$$

where the differential coefficients on the left-hand side are total, and those on the right-hand side partial.

Hence, using the identities (4), we get

$$\frac{dc_1}{dt} = -\frac{dc_1}{dp}\frac{dK}{dq} + \frac{dc_1}{dq}\frac{dK}{dp} + \ldots\ldots\ldots \qquad (5),$$

with similar expressions for $\frac{dc_2}{dt}$, &c.

If K be given as a function of p, q, &c. and t, we have $\frac{dc_1}{dt}$, &c. expressed as functions of p, q, &c. and t. Joining these equations to those marked (1) we find c_1, c_2 ... as functions of t. If K be given as a function of c_1, c_2, ... and t we may continue thus,

$$\frac{dK}{dp}=\frac{dK}{dc_1}\frac{dc_1}{dp}+\frac{dK}{dc_2}\frac{dc_2}{dp}+\dots,$$

$$\frac{dK}{dq}=\frac{dK}{dc_1}\frac{dc_1}{dq}+\frac{dK}{dc_2}\frac{dc_2}{dq}+\dots.$$

Substituting in the expression for $\frac{dc_1}{dt}$, we get

$$\frac{dc_1}{dt}=\Sigma\left[\frac{dc_1}{dq}\frac{dc_2}{dp}-\frac{dc_1}{dp}\frac{dc_2}{dq}\right]\frac{dK}{dc_2}$$
$$+\Sigma\left[\frac{dc_1}{dq}\frac{dc_3}{dp}-\frac{dc_1}{dp}\frac{dc_3}{dq}\right]\frac{dK}{dc_3}$$
$$+\dots,$$

where the Σ means summation for all values of p, q, viz. p_1, q_1, p_2, q_2, &c.

Since by hypothesis c_1, c_2, ... are supposed expressed as functions of p_1, q_1, &c. and t, these coefficients may be found by simple differentiation. It will, of course, be more convenient to express them in terms of c_1, c_2, &c. and t by substituting for p_1, q_1, &c. their values given by the integrals (1).

176. On effecting this substitution it will be found that t disappears from the expressions. This may be proved as follows. Let A be any coefficient, so that

$$A=\Sigma\left[\frac{dc_1}{dq}\frac{dc_2}{dp}-\frac{dc_1}{dp}\frac{dc_2}{dq}\right],$$

we have to prove that A being regarded as a function of p_1, q_1, &c. and t, the total differential coefficient $\frac{d.A}{dt}$ is zero. Now

$$\frac{d.A}{dt}=\frac{dA}{dt}+\frac{dA}{dp}p'+\frac{dA}{dq}q'+\dots.$$

The letters p_1, q_1, &c. enter into the expression for A only through c_1 and c_2. Let us consider only the part of $\frac{d.A}{dt}$ due to the variation of c_1, then the part due to the variation of c_2 may be found by interchanging c_1 and c_2, and changing the sign of the whole. The complete value of $\frac{d.A}{dt}$ is the sum of these two parts.

The part of $\frac{d.A}{dt}$ due to the variation of c_1 is

$$\Sigma\left[\frac{dc_2}{dp}\left\{\frac{d}{dq}\frac{dc_1}{dt}-\frac{d^2c_1}{dp\,dq}\frac{dH}{dq}+\frac{d^2c_1}{dq^2}\frac{dH}{dp}+\ldots\right\}\right.$$
$$\left.-\frac{dc_2}{dq}\left\{\frac{d}{dp}\frac{dc_1}{dt}-\frac{d^2c_1}{dp^2}\frac{dA}{dq}+\frac{d^2c_1}{dp\,dq}\frac{dH}{dp}+\ldots\right\}\right].$$

If we substitute for $\frac{dc_1}{dt}$ its value given by the identity (4), we get

$$\Sigma\left[\frac{dc_2}{dp}\left\{\frac{dc_1}{dp}\frac{d^2H}{dq^2}-\frac{dc_1}{dq}\frac{d^2H}{dp\,dq}\right\}\right.$$
$$\left.-\frac{dc_2}{dq}\left\{\frac{dc_1}{dp}\frac{d^2H}{dp\,dq}-\frac{dc_1}{dq}\frac{d^2H}{dp^2}\right\}\right].$$

If we now interchange c_1 and c_2 we get the same result. Hence when the two parts of $\frac{d.A}{dt}$ are added together, the signs being opposite, the sum is zero.

177. Let the expression

$$\Sigma\left[\frac{dc_1}{dq}\frac{dc_2}{dp}-\frac{dc_1}{dp}\frac{dc_2}{dq}\right],$$

where the Σ means summation for all the values of p, q, be represented shortly by (c_1, c_2). Then in any dynamical problem if K be the disturbing function, the variations of the parameters $c_1, c_2, \ldots$ are given by

$$\frac{dc_1}{dt} = (c_1, c_2)\frac{dK}{dc_2} + (c_1, c_3)\frac{dK}{dc_3} + \ldots,$$

where all the coefficients are functions of the parameters only and not of t.

This equation may be greatly simplified by a proper choice of the constants $c_1, c_2, \ldots$. In the *Mécanique Analytique* of Lagrange, it is shown that if the constants chosen be the *initial* values of $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$, viz. $\alpha, \beta, \gamma, \ldots$ and $\lambda, \mu, \nu, \ldots$ respectively, then the equations become

$$\left.\begin{array}{l} \dfrac{d\alpha}{dt} = -\dfrac{dK}{d\lambda}, \quad \dfrac{d\beta}{dt} = -\dfrac{dK}{d\mu}, \text{ \&c.} \\ \dfrac{d\lambda}{dt} = \dfrac{dK}{d\alpha}, \quad \dfrac{d\mu}{dt} = \dfrac{dK}{d\beta}, \text{ \&c.} \end{array}\right\}.$$

It is assumed in the demonstration that K is a function of $q_1, q_2, \ldots$ only. This simplification has been extended by Sir W. Hamilton and Jacobi to other cases, but for this we must refer the reader to books which treat on theoretical dynamics.

178. It follows from the investigation in Art. 176, that if two integrals of a dynamical problem be found, viz.

$$c_1 = \alpha, \quad c_2 = \beta,$$

where c_1 and c_2 stand for some functions of $p_1, q_1, p_2, q_2, \ldots$ and t, and α and β are constants, then (c_1, c_2) is also constant. So that $(c_1, c_2) = \gamma$, where γ is a constant, is either a third integral of the equations of motion or is an identity. If it is an integral it may be either a new integral or one derivable from the two c_1 and c_2 already found.

The Principle of Least Action.

179. DEF. Let v be the velocity of any particle whose mass is m and let ds be an element of the path traced out by it, then the value of $m\int v ds$ taken between any limits of s is called the "action" of the particle in passing from one position to the other.

Let a system of bodies be subject only to their mutual actions or to forces tending to fixed centres. Then in moving from any one given position to another their paths and velocities are such that the sum of the "actions" of all the particles is less than if the bodies had been constrained to take any other courses by which they could be brought from their initial to their final positions.

This constraint may be effected by the introduction of new geometrical relations or in any other manner. But in all cases it is supposed that no given geometrical relations are violated and that the same law of vis viva holds throughout all the different motions.

Since $m\int v ds = m\int v^2 dt$, it is clear that the whole action of the system is what Hamilton has represented by the symbol V. Now let the system be constrained to take any other course, then by the same reasoning as before, see Art. 171,

$$\delta V = \Sigma \frac{dT}{dq'} \delta q - \Sigma \frac{dT_0}{da'} \delta a + t\delta h.$$

This new course, by hypothesis, is such that the system does really go from the same initial to the same final positions in such a manner that the constant h of vis viva is unaltered.

Hence $$\delta a = 0,\ \delta q = 0,\ \delta h = 0,$$
$$\therefore\ \delta V = 0.$$

That is, V is either a maximum or a minimum. It cannot be a maximum, since by causing the bodies to take circuitous routes we may make V as large as we please. Hence V is a minimum.

If the system* be acted on by no external forces and be such that the vis viva is constant, we have

$$V = \int_0^t T dt = T \cdot t.$$

Hence t is a minimum, that is the system will pass from one given position to another in less time than if it were compelled to take any other route than that which in accordance with the laws of motion it does actually choose for itself.

If there be but one particle constrained to rest on a given surface, and if it be subject to the action of no force, then its velocity is constant and the particle in passing from one given position on the surface to another will describe the shortest line that can be drawn between them on the given surface.

On Impulsive Forces.

180. The Lagrangian equations of motion may be applied to impulsive forces. We may regard an impulse as the limit of a large finite force acting for a very short time. If t_0 and t_1 be the times at which the force begins and ceases to act, the required equations may be found by integrating the equation

$$\frac{d}{dt}\frac{dT}{dq'} - \frac{dT}{dq} = \frac{dU}{dq}$$

between the limits $t = t_0$ and $t = t_1$.

The integral of the first term† is $\left[\frac{dT}{dq'}\right]_{t_0}^{t_1}$, which is the difference between the initial and final values of $\frac{dT}{dq'}$. The integral of the second term is zero, for consider any term $Aq_1'q_2'$ of the expression for $\frac{dT}{dq}$, where A is a function of $q_1, q_2 \dots q_n$. The interval of impact is so short that the change in position of the particles may be neglected. We have therefore

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* See Earnshaw's *Dynamics*.

† See a paper by Prof. Niven in the *Mathematical Messenger* for May, 1867.

$$\int_{t_0}^{t_1} Aq_1' q_2' \, dt = A \int_{t_0}^{t_1} q_1' q_2' \, dt.$$

Let ω_1, ω_2 be the greatest values of q_1', q_2' during the impact, which are clearly both finite. Then this integral is less than $A\omega_1 \omega_2 (t_1 - t_0)$ which vanishes when in the limit t_1 becomes equal to t_0. Also if δU_1 be the virtual moment of the impulsive forces we shall have on the same suppositions as before $\frac{dU_1}{dq_1} = \int_{t_0}^{t_1} \frac{dU}{dq} dt$. Hence we may write the Lagrangian equations of impulses in the form

$$\left[\frac{dT}{dq'}\right]_{t_0}^{t_1} = \frac{dU_1}{dq}.$$

The principle of Virtual Velocities applied to impulsive forces gives the general equation

$$\Sigma m \{(x_1' - x_0') \delta x + (y_1' - y_0') \delta y + \&c.\} = \Sigma \{X \delta x + Y \delta y + \&c.\},$$

where (x_0', y_0', z_0') (x_1', y_1', z_1') are the resolved velocities of the particle m parallel to the axes just before and after the action of the impulsive forces whose components are X, Y, Z. From this we may at once deduce the Lagrangian equations of impulses. For by referring to Art. 165, we see that

$$x' \delta x = \frac{1}{2} \frac{d(x'^2)}{d\theta'} \delta\theta + \&c.$$

with similar expressions for $y' \delta y$, $z' \delta z$. Hence the left-hand side of the above equation is

$$\left[\Sigma m \left\{\frac{dT}{d\theta'} \delta\theta + \frac{dT}{d\phi'} \delta\phi + \dots\right\}\right]_{t_0}^{t_1},$$

and the right-hand side is evidently

$$\frac{dU_1}{d\theta} \delta\theta + \frac{dU_1}{d\phi} \delta\phi + \dots$$

Comparing these we get the equations of motion.

181. **Example.** As an illustration of the use of these equations let us consider the example already discussed in Art. 85.

Let x be the altitude of the centre of gravity of the rhombus at any time, then x and α may be taken as the independent variables.

We have

$$T = 2\{x'^2 + (k^2 + a^2)\alpha'^2\}.$$

Let P be the impulsive action between the rhombus and the plane, then the virtual moment of the impulsive forces is

$$\begin{aligned}\delta U &= P\delta(x - 2a\cos\alpha)\\ &= P\delta x + 2a\sin\alpha\, P\delta\alpha.\end{aligned}$$

The Lagrangian equations are therefore

$$\left.\begin{aligned}&4(x_1' - x_0') = P\\ &4(k^2 + a^2)(\alpha_1' - \alpha_0') = 2aP\sin\alpha\end{aligned}\right\}.$$

Now the initial and final values of x' are

$$x_0' = -V,\ x_1' = -2a\sin\alpha\omega;$$

those of α' are $\qquad \alpha_0' = 0, \quad \alpha_1' = \omega.$

Hence eliminating P we have

$$\omega' = \frac{3}{2}\frac{V}{a}\frac{\sin\alpha}{1 + 3\sin^2\alpha},$$

the same result as before.

182. *To determine the change in the vis viva of a moving system produced by any collisions between the bodies or by any explosions.* (Carnot's Theorem.)

Let $v_x, v_y, v_z, v_x', v_y', v_z'$ be the resolved parts of the velocities of any particle m of the system before and after the impulse.

Then by D'Alembert's Principle the momenta

$$m(v_x' - v_x), \quad m(v_y' - v_y), \quad m(v_z' - v_z),$$

being reversed and taken throughout the whole system, are in equilibrium with the forces of the impulse. But these last are themselves in equilibrium. Hence the former set are also in equilibrium. Therefore by Virtual Velocities,

$$\Sigma m \{(v_x' - v_x)\,\delta x + (v_y' - v_y)\,\delta y + (v_z' - v_z)\,\delta z\} = 0,$$

where δx, δy, δz are any small arbitrary displacements of the particles impinging on each other, which are consistent with the geometrical conditions of the system during the time of action of the impulse.

During the impact, it is one geometrical condition that the particles impinging on each other have no velocity of separation normal to the common surface of the bodies of which they form a part.

First. Let the bodies be devoid of elasticity. Then the above geometrical condition will hold just *after* the moment of greatest compression as well as during the impact. Hence we can put

$$\delta x = v_x'\delta t, \quad \delta y = v_y'\delta t, \quad \delta z = v_z'\delta t.$$

The equation now becomes

$$\Sigma m \{(v_x' - v_x)\, v_x' + (v_y' - v_y)\, v_y' + (v_z' - v_z)\} = 0\,;$$

$$\therefore\ \Sigma m\, (v_x'^2 + v_y'^2 + v_z'^2) = \Sigma m\, (v_x v_x' + v_y v_y' + v_z v_z').$$

This may be put into the form

$$\Sigma m\, (v_x'^2 + v_y'^2 + v_z'^2) - \Sigma m\, (v_x^2 + v_y^2 + v_z^2)$$
$$= -\Sigma m \{(v_x' - v_x)^2 + (v_y' - v_y)^2 + (v_z' - v_z)^2\}.$$

Therefore in the impact of inelastic bodies vis viva is always lost.

Secondly. Let an explosion take place in any body of the system. Then the geometrical equation above spoken of will hold just before the impulse begins as well as during the explosion, but it will not hold after the particles of the body have separated. Hence we must now put

$$\delta x = v_x\delta t, \quad \delta y = v_y\delta t, \quad \delta z = v_z\delta t.$$

As before, we have

$$\Sigma m\, (v_x v_x' + v_y v_y' + v_z v_z') = \Sigma m\, (v_x^2 + v_y^2 + v_z^2),$$

and

$$\Sigma m\,(v_x'^2 + v_y'^2 + v_z'^2) - \Sigma m\,(v_x^2 + v_y^2 + v_z^2)$$
$$= + \Sigma m\,\{(v_x' - v_x)^2 + (v_y' - v_y)^2 + (v_z' - v_z)^2\}.$$

Therefore in cases of explosion vis viva is always gained.

Thirdly. Let the particles of the system be perfectly elastic. Then the whole action consists of two parts, a force of compression as if the particles were inelastic, and a force of restitution of the nature of an explosion. The circumstances of these two forces are exactly equal and opposite to each other. By examining these two expressions it is easy to see that the vis viva lost in the compression is exactly balanced by the vis viva gained in the restitution.

EXAMPLES.

1. A uniform rod is moving on a horizontal table about one extremity, and driving before it a particle of mass equal to its own, which starts from rest indefinitely near to the fixed extremity; show that when the particle has described a distance r along the rod, its direction of motion makes with the rod an angle

$$\tan^{-1}\frac{k}{\sqrt{r^2+k^2}}.$$

2. A thin uniform smooth tube is balancing horizontally about its middle point, which is fixed; a uniform rod such as just to fit the base of the tube is placed end to end in a line with the tube, and then shot into it with such a horizontal velocity that its middle point shall only just reach that of the tube; supposing the velocity of projection to be known, find the angular velocity of the tube and rod at the moment of the coincidence of their middle points.

Result. If m be the mass of the rod, m' that of the tube, and $2a$, $2a'$ their respective lengths, v the velocity of the rod's projection, ω the required angular velocity, then

$$\omega^2 = \frac{3mv^2}{ma^2 + m'a'^2}.$$

3. A fine circular tube, carrying within it a heavy particle, is set revolving about a vertical diameter. Show that the difference of the squares of the absolute velocities of the particle at any two given points of the tube equidistant from the axis is the same for all initial velocities of the particle and tube.

4. A screw of Archimedes is capable of turning freely about its axis, which is fixed in a vertical position: a heavy particle is placed at the top of the tube and runs down through it; determine the whole angular velocity communicated to the screw.

Result. Let n be the ratio of the mass of the screw to that of the particle, $\alpha =$ the angle the tangent to the screw makes with the horizon, h the height descended by the particle. Then the angular velocity generated is

$$= \sqrt{\frac{2gh\cos^2\alpha}{a^2(n+1)(n+\sin^2\alpha)}}.$$

5. A cone of mass m and vertical angle 2α can move freely about its axis, and has a fine smooth groove cut along its surface so as to make a constant angle β with the generating lines of the cone. A heavy particle of mass P moves along the groove under the action of gravity, the system being initially at rest with the particle at a distance c from the vertex. Show that if θ be the angle through which the cone has turned when the particle is at any distance r from the vertex, then

$$\frac{mk^2 + Pr^2\sin^2\alpha}{mk^2 + Pc^2\sin^2\alpha} = \epsilon^{2\theta\sin\alpha\,.\,\cot\beta},$$

k being the radius of gyration of the cone about its axis.

6. Two equal beams connected by a hinge at their centres of gravity so as to form an X are placed symmetrically on two smooth pegs in the same horizontal line, the distance between which is b. Show that, if the beams be perpendicular to each other at the commencement of the motion, the velocity of their centre of gravity when in the line joining the

pegs is equal to $\sqrt{\dfrac{b^3g}{b^2+4k^2}}$, where k is the radius of gyration of either beam about a line perpendicular to it through its centre of gravity.

7. A lamina of any form rolls on a perfectly rough straight line under the action of no forces; prove that the velocity v of the centre of gravity G is given by $v^2=c^2\dfrac{r^2}{r^2+k^2}$, where r is the distance of G from the point of contact, and k is the radius of gyration of the body about an axis through G perpendicular to its plane, and c is some constant.

8. If an elastic string, whose natural length is that of a uniform rod, be attached to the rod at both ends and suspended by the middle point, prove by means of vis viva that the rod will sink until the strings are inclined to the horizon at an angle θ, which satisfies the equation

$$\cot^3\frac{\theta}{2}-\cot\frac{\theta}{2}-2n=0,$$

where the tension of the string, when stretched to double its length, is n times the weight.

If the string be suspended by a point, not in the middle, write down the equation of vis viva.

9. A circular wire ring, carrying a small bead, lies on a smooth horizontal table; an elastic thread the natural length of which is less than the diameter of the ring, has one end attached to the bead and the other to a point in the wire; the bead is placed initially so that the thread coincides very nearly with a diameter of the ring; find the vis viva of the system when the string has contracted to its original length.

10. A straight tube of given length is capable of turning freely about one extremity in a horizontal plane, two equal particles are placed at different points within it at rest, an angular velocity is given to the system, determine the velocity of each particle on leaving the tube.

11. A rough plane rotates with uniform angular velocity n about a horizontal axis which is parallel to it but not in it. A heavy sphere of radius a being placed on the plane when in a horizontal position, rolls down it under the action of gravity. If the centre of the sphere be originally in the plane containing the moving axis and perpendicular to the moving plane, and if x be its distance from this plane at a subsequent time t before the sphere leaves the plane, then

$$x = A\left(e^{\sqrt{\frac{5}{7}}nt} - e^{-\sqrt{\frac{5}{7}}nt}\right) - \frac{5}{12}\frac{g}{n^2}\sin nt,$$

$$\text{where } A = \frac{1}{24\sqrt{35}}\left(\frac{35g}{n^2} - 84a - 60c\right),$$

c being the distance from the axis to the plane measured upwards.

12. Two particles of masses m, m' are connected by an inelastic string of length a. The former is placed in a smooth straight groove and the latter is projected in a direction perpendicular to the groove with a velocity V. Prove that the particle m will oscillate through a space $\dfrac{2am'}{m+m'}$, and if m be large compared with m' the time of oscillation is nearly $\dfrac{\pi a}{V}\left(1 - \dfrac{m'}{4m}\right)$.

13. The extremities of the axis of a uniform solid of revolution symmetrical with respect to its equator, slide freely on two smooth intersecting rods at right angles to each other, one of which is fixed and vertical and the other revolves freely about it, their point of intersection being fixed. An angular velocity λ about the vertical rod is impressed on the system while the solid spins with an angular velocity n about its axis, which is initially inclined at an angle α to the vertical. Prove that

$$(Ma^2 + A)\sin^2\theta\frac{d\phi}{dt} + Cn\cos\theta = (Ma^2 + A)\lambda\sin^2\alpha + Cn\cos\alpha,$$

$$(Ma^2 + A)\left\{\frac{d^2\theta}{dt^2} - \left(\frac{d\phi}{dt}\right)^2\cos\theta\sin\theta\right\} + Cn\frac{d\phi}{dt}\sin\theta = -Mga\sin\theta,$$

M being the mass of the solid, $2a$ the length of its axis; C, A its moments of inertia about its axis of figure and about an equatorial diameter respectively, θ the inclination of the axis to the vertical, and ϕ the angle through which the horizontal rod has revolved after a time t.

14. The centre C of a circular wheel is fixed and the rim is constrained to roll in a uniform manner on a perfectly rough horizontal plane so that the plane of the wheel makes a constant angle α with the vertical. Round the circumference there is a uniform smooth canal of very small section, and a heavy particle which just fits the canal can slide freely along it under the action of gravity. If m be the particle, B the point where the wheel touches the plane and $\theta = \angle BCm$, and if n be the angular rate at which B describes the circular trace on the horizontal plane, prove that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{a}\cos\alpha\cos\theta - n^2\cos^2\alpha\cos^2\theta + \text{const.}$$

where a is the radius of the wheel.

15. A regular hexagonal inelastic prism each side of which is r feet, is at rest on a perfectly rough inclined plane inclined at an angle 30° to the horizon, with its axis horizontal. Supposing it to be just disturbed from its position of rest, show that the velocity V of its centre of gravity after the nth impact is given by

$$V^2 = \frac{17gr}{14}\left\{1 - \left(\frac{11}{17}\right)^{2n}\right\}.$$

16. The extremities of a uniform heavy beam of length $2a$ slide on a smooth wire in the form of the curve whose equation is $r = a(1 - \cos\theta)$ the prime radius being vertical and the vertex of the curve downwards. Prove that if the beam be placed in a vertical position and displaced with a velocity just sufficient to bring it into a horizontal position

$$\tan\theta = \frac{1}{2}\left\{e^{\sqrt{\frac{3g}{2a}}t} - e^{-\sqrt{\frac{3g}{2a}}t}\right\},$$

where θ is the angle through which the rod has turned after a time t.

17. A rigid body whose radius of gyration about G the centre of gravity is k, is attached to a fixed point C by a string fastened to a point A on its surface. $CA\ (=b)$ and $AG\ (=a)$ are initially in one line, and to G is given a velocity V at right angles to that line. No impressed forces are supposed to act, and the string is attached so as always to remain in one right line. If θ be the angle between AG and AC at time t, show that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{V^2}{b^2}\,\frac{k^2 - 4\,ab\sin^2\frac{\theta}{2}}{k^2 + a^2\sin^2\theta}$$

and if the amplitude of θ, *i.e.* $2\sin^{-1}\frac{k}{2\sqrt{ab}}$ be very small, the period is $\frac{2\pi bk}{V\sqrt{a(a+b)}}$.

18. A fine weightless string having a particle at one extremity is partially coiled round a hoop which is placed on a smooth horizontal plane, and is capable of motion about a fixed vertical axis through its centre. If the hoop be initially at rest and the particle be projected in a direction perpendicular to the length of the string, and if s be the portion of the string unwound at any time t, then

$$s^2 - b^2 = \frac{\mu}{m+\mu}\,V^2 t^2 + 2\,Vat,$$

where b is the initial value of s, m and μ the masses of the hoop and particle, a the radius of the hoop and V the velocity of projection.

19. A square formed of four similar uniform rods jointed freely at their extremities is laid upon a smooth horizontal table, one of its angular points being fixed: if angular velocities ω, ω' in the plane of the table be communicated to the two sides containing this angle, show that the greatest value of the angle (2α) between them is given by the equation

$$\cos 2\alpha = -\frac{5}{6}\,\frac{(\omega-\omega')^2}{\omega^2+\omega'^2}.$$

20. Two particles of masses m, m' lying on a smooth horizontal table are connected by an inelastic string extended to its full length and passing through a small ring on the table. The particles are at distances a, a' from the ring and are projected with velocities v, v' at right angles to the string. Prove that if $mv^2a^2 = m'v'^2a'^2$, their second apsidal distances from the ring will be a', a respectively.

21. If a uniform thin rod PQ move in consequence of a primitive impulse between two smooth curves in the same plane, prove that the square of the angular velocity varies inversely as the difference between the sum of the squares of the normals OP, OQ to the curves at the extremities of the rods, and $\frac{5}{12}$ of the square of the whole length of the rod.

22. A small bead can slide freely along an equiangular spiral of equal mass which can turn freely about its pole as a fixed point. A centre of repulsive force F is situated in the pole and acts on the particle. If the system start from rest when the particle is at a distance a, show that the angular velocity of the spiral when the particle is at a distance k from the pole is $\sqrt{\dfrac{\int_a^k F dr}{mk^2}}$ where mk^2 is the moment of inertia of the spiral about its pole.

23. The extremities of a uniform beam of length $2a$, slide on two slender rods without inertia, the plane of the rods being vertical, their point of intersection fixed and the rods inclined at angles $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ to the horizon. The system is set rotating about the vertical line through the point of intersection of the rods with an angular velocity ω, prove that if θ be the inclination of the beam to the vertical at the time t and α the initial value of θ

$$4\left(\frac{d\theta}{dt}\right)^2 + \frac{(3\cos^2\alpha + \sin^2\alpha)^2}{3\cos^2\theta + \sin^2\theta}\omega^2 = (3\cos^2\alpha + \sin^2\alpha)\,\omega^2$$
$$+ \frac{6g}{a}(\sin\alpha - \sin\theta).$$

CHAPTER VIII.

SMALL OSCILLATIONS.

183. WHEN a system of bodies admits of only one independent motion, and is making small oscillations about some mean position, it is in general our object to reduce the equations of motion to the form

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = c,$$

where x is some small quantity which determines the position of the system at the time t, and a, b, c are some constants. This reduction is effected by neglecting the squares of the small quantity x. The integral of this equation is known to be

$$x = \frac{c}{b} + Ae^{-\frac{at}{2}} \sin\left\{\sqrt{b - \frac{a^2}{4}}\,t + B\right\},$$

where A and B are two undetermined constants which depend on the initial conditions of motion. The physical interpretation of this equation is not difficult. It represents an oscillatory motion, and the period of a complete oscillation is $\dfrac{2\pi}{\sqrt{b - \dfrac{a^2}{4}}}$. The central position about which the system oscillates is determined by $x = \dfrac{c}{b}$. The extent of the oscillation on either side of this central position is determined by $x = \dfrac{c}{b} \pm Ae^{-\frac{at}{2}}$, and depends on the initial conditions. If a

be positive the extent of the oscillations continually decreases and the subsequent arcs will be in geometrical progression. If $b-\frac{a^2}{4}$ be negative, the sine must be replaced by its exponential value, and the integral becomes

$$x=\frac{c}{b}+A'e^{\left(-\frac{a}{2}+\sqrt{\frac{a^2}{4}-b}\right)t}+B'e^{\left(-\frac{a}{2}-\sqrt{\frac{a^2}{4}-b}\right)t},$$

where A', B' are two undetermined constants. The motion is now no longer oscillatory. If $b-\frac{a^2}{4}=0$, the integral takes a different form and we have

$$x=\frac{c}{b}+(A''t+B'')\,e^{-\frac{at}{2}},$$

where A'', B'' are two undetermined constants.

In all these cases, except that in which a and $b-\frac{a^2}{4}$ are both positive, the magnitude of x will continually increase. The terms depending on x^2 which have been neglected in forming the equation will then become large. It is possible that they may alter the whole character of the motion. In such cases the equilibrium of the body is called unstable, and these equations can represent only the nature of the motion with which the system *begins* to move from its state of rest.

First Method.

184. When the system under consideration consists of a single body, there is a simple method of forming the equation of motion which is sometimes of great use.

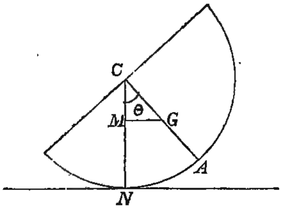
It has been shown in Art. 92, that if we neglect the squares of small quantities we may take moments about the instantaneous centre as a fixed centre. Usually the unknown reactions will be such that their lines of action will pass through this point, their moments will then be zero and thus we shall have an equation containing only known quantities.

Since the body is supposed to be turning about the instantaneous centre as a point fixed for the moment, the direction of motion of any point of the body is perpendicular to the straight line joining it to the centre. Conversely when the directions of motion of two points of the body are known, the position of the instantaneous centre can be found. For if we draw perpendiculars at these points to their directions of motion, these perpendiculars must meet in the instantaneous centre of rotation.

185. Ex. 1. *A hemisphere performs small oscillations on a perfectly rough horizontal plane: find the motion.*

Let C be the centre, G the centre of gravity of the hemisphere, N the point of contact with the rough plane.

Let the radius $= a$, $CG = c$, $\theta = \angle NCG$.



Here the point N is the centre of instantaneous rotation, because the plane being perfectly rough, sufficient friction will be called into play to keep N at rest. Hence taking moments about N

$$(k^2 + GN^2)\frac{d^2\theta}{dt^2} = -gc \,.\, \sin\theta.$$

Since we can put $GN = a - c$ in the small terms, this reduces to

$$\{k^2 + (a-c)\}^2 \frac{d^2\theta}{dt^2} + gc \,.\, \theta = 0.$$

Therefore the time of a small oscillation is

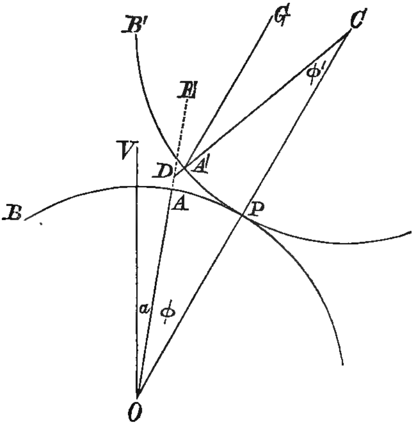
$$= 2\pi\sqrt{\frac{k^2 + (a-c)^2}{cg}}.$$

It is clear that $k^2 + c^2 = (\text{rad.})^2$ of gyration about $C = \frac{2}{5} a^2$ and $c = \frac{3}{8} a$.

If the plane had been smooth, M would have been the instantaneous axis, GM being the perpendicular on CN. For the motion of N is in a horizontal direction, because the sphere remains in contact with the plane, and the motion of G is vertical by Art. 46. Hence the two perpendiculars GM, NM meet in the instantaneous axis. By reasoning similar to the above the time will be found to be $2\pi\sqrt{\frac{k^2}{cg}}$.

186. Ex. 2. *A cylindrical surface of any form rests in stable equilibrium on another perfectly rough cylindrical surface, the axes of the cylinders being parallel. A small disturbance being given to the upper surface, find the time of a small oscillation.*

Let BAP, $B'A'P$ be sections of the cylinders perpendicular to their axes. Let OA, CA' be normals at those points A, A' which before disturbance were in contact, and



let α be the angle AO makes with the vertical. Let OPC be the common normal at the time t. Let G be the centre of gravity of the moving body, then before disturbance $A'G$ was vertical. Let $A'G = r$.

Now we have only to determine the time of oscillation when the motion decreases without limit. Hence the arcs AP, $A'P$ will be ultimately zero, and therefore C and O may be taken as the centres of curvature of AP, $A'P$. Let $\rho = OA$, $\rho' = CA'$, and let the angles AOP, $A'CP$ be denoted by ϕ, ϕ' respectively.

Let θ be the angle turned round by the body in moving from the position of equilibrium into the position $B'A'P$. Then since before disturbance, $A'C$ and AO were in the same straight line, we have $\theta = \angle CDE = \phi + \phi'$, where CA' meets OAE in D.

Also since one body rolls on the other, the arc $AP =$ arc $A'P$, $\therefore \rho\phi = \rho'\phi'$, $\therefore \phi = \dfrac{\rho'}{\rho + \rho'}\theta$.

Again, in order to take moments about P, we require the horizontal distance of G from P; this may be found by projecting the broken line $PA' + A'G$ on the horizontal. The projection of $PA' = PA' \cos(\alpha + \theta) = \rho\phi \cos\alpha$ when we neglect the squares of small quantities. The projection of $A'G$ is $r\theta$. Thus the horizontal distance required is

$$\left(\frac{\rho\rho'}{\rho + \rho'}\cos\alpha - r\right)\theta.$$

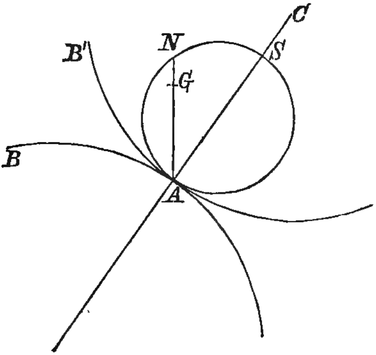
If k be the radius of gyration about the centre of gravity, the equation of motion is

$$(k^2 + GA^2)\frac{d^2\theta}{dt^2} = -g\theta\left(\frac{\rho\rho'}{\rho + \rho'}\cos\alpha - r\right).$$

If L be the length of the simple equivalent pendulum, we have

$$\frac{k^2 + r^2}{L} = \frac{\rho\rho'}{\rho + \rho'}\cos\alpha - r.$$

187. Along the common normal at the point of contact A of the two cylindrical surfaces measure a length $AS = s$ where $\frac{1}{s} = \frac{1}{\rho} + \frac{1}{\rho'}$, and describe a circle on AS as diameter. Let AG, produced if necessary, cut this circle in N. Then



$GN = s \cos \alpha - r$, the positive direction being from N towards A. The length L of the simple equivalent pendulum is given by the formula

$$\frac{k^2 + GA^2}{L} = GN.$$

It is clear from this formula, that if G* lie without the circle L is negative and the equilibrium is unstable, if within L is positive and equilibrium is stable.

188. Ex. 3. *Two points* A, B *of a body are constrained to describe given curves, and the body is in equilibrium under*

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* Let R be the radius of curvature of the path traced out by G as the one cylinder rolls on the other, then we know that $R = -\frac{AG^2}{NG}$, so that all points within the circle described on AS as diameter are describing curves whose concavity is turned towards A, while those within the circle are describing curves whose convexity is turned towards A. It is then clear that the equilibrium is stable, unstable, or neutral, according as the centre of gravity lies within, without, or on the circumference of the circle.

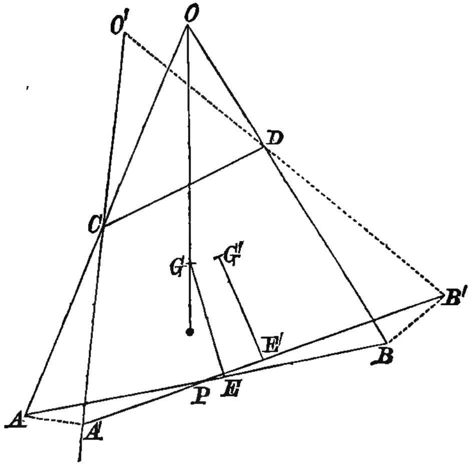
the action of gravity. A small disturbance being given, find the time of an oscillation.

Let C, D be the centres of curvature of the given curves at the two points A, B. Let AC, BD meet in O. Let G be the centre of gravity of the body, GE a perpendicular on AB. Then in the position of equilibrium OG is vertical. Let i, j be the angles CA, BD make with the vertical, and let α be the angle AOB.

Let A', B'...denote the positions into which A, B...have been moved when the body has been turned through an angle θ. Let $ACA' = \phi$, $BDB' = \phi'$. Since the body may be brought from the position AB into the position $A'B'$ by turning it about O through an angle θ, we have $\dfrac{CA \, . \, \phi}{OA} = \dfrac{BD \, . \, \phi'}{OB} = \theta$. Also GG' is ultimately perpendicular to OG, and we have $GG' = OG \, . \, \theta$.

Also let x, y be the projections of OO' on the horizontal and vertical through O. Then by projections

$$x \cos j + y \sin j = \text{distance of } O' \text{ from } OD$$
$$= OD \, . \, \phi',$$



$$x\cos i - y\sin i = \text{distance of } O' \text{ from } OC$$
$$= OC\,.\,\phi;$$
$$\therefore x = \frac{OD\sin i\,\phi' + OC\,.\sin j\,.\,\phi}{\sin\alpha}.$$

Now taking moments about O' as the centre of instantaneous rotation, we have

$$(k^2 + \overline{OG}^2)\,\frac{d^2\theta}{dt^2} = -g\,.\,(GG' + x)$$
$$= -g\theta\left(OG + \frac{OD\,.\,OB}{BD}\,.\,\frac{\sin i}{\sin\alpha} + \frac{OC\,.\,OA}{CA}\,\frac{\sin j}{\sin\alpha}\right),$$

where k is the radius of gyration about the centre of gravity.

Hence if L be the length of the simple equivalent pendulum, we have

$$\frac{k^2 + OG^2}{L} = OG + \frac{OD\,.\,OB}{BD}\,.\,\frac{\sin i}{\sin\alpha} + \frac{OC\,.\,OA}{AC}\,.\,\frac{\sin j}{\sin\alpha}\,.$$

COR. If the given curves, on which the points A, B are constrained to move, be straight lines, the centres of curvature C and D are at infinity. In this case, we may put $\dfrac{OD}{BD} = -1$, $\dfrac{OC}{AC} = -1$, and the expression becomes

$$\frac{k^2 + OG^2}{L} = OG - OB\,.\,\frac{\sin i}{\sin\alpha} - OA\,.\,\frac{\sin j}{\sin\alpha}\,.$$

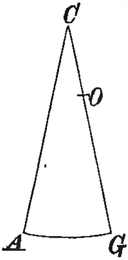
If OA and OB be at right angles, this takes the simple form

$$\frac{k^2 + OG^2}{L} = OG - 2OF,$$

where F is the projection on OG of the middle point of AB.

189. When the body is capable of only one independent motion, the problem of finding the time of a small oscillation is really the same as that of finding the radius of curvature of the path of the centre of gravity.

Let A be the position of the centre of gravity of the body when it is in its position of equilibrium, G the position of the centre of gravity at the time t. Then since in equilibrium the altitude of the centre of gravity is a maximum or minimum, the tangent at A to the curve AG is horizontal. Let the normal GC to the curve at G meet the normal at A in C. Then when the oscillation becomes indefinitely small C is the centre of curvature of the curve at A. Let $AG = s$, the angle $ACG = \psi$, and let R be the radius of curvature of the curve at A.



Let θ be the angle turned round by the body in moving from the position of equilibrium into the position in which the centre of gravity is at G; then $\frac{d\theta}{dt}$ is the angular velocity of the body. Since G is moving along the tangent at G, the centre of instantaneous rotation lies in the normal GC, at such a point O, that $OG\frac{d\theta}{dt} = \text{vel. of } G = \frac{ds}{dt}$, $\therefore\ GO = \frac{ds}{d\theta}$.

Let Mk^2 be the moment of inertia of the body about its centre of gravity, then taking moments about O, we have

$$(k^2 + OG^2)\frac{d^2\theta}{dt^2} = -g\,.\,OG\sin\psi.$$

Now ultimately when the angle θ is indefinitely small $\frac{\psi}{\theta} = \frac{d\psi}{d\theta} = \frac{OG}{R}$; $\therefore$ the equation of motion becomes

$$(k^2 + OG^2)\frac{d^2\theta}{dt^2} = -g\frac{OG^2}{R}\,.\,\theta.$$

Hence if L be the length of the simple equivalent pendulum we have

$$L = \left(1 + \frac{k^2}{OG^2}\right) R.$$

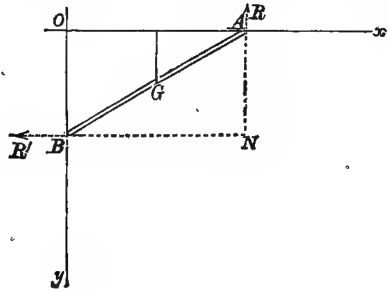
Second Method.

190. Let the general equations of motion of all the bodies be formed. If the position about which the system oscillates be known, some of the quantities involved will be small. The squares and higher powers of these may be neglected, and all the equations will become linear. If the unknown reactions be then eliminated the resulting equations may be easily solved.

If the position about which the system oscillates be unknown, it is not necessary to solve the statical problem first. We may by one process determine the positions of rest, ascertain whether they are static or not, and find the time of oscillation. The method of proceeding will be best explained by an example.

191. Ex. 1. *The ends of a uniform heavy rod* AB *of length* 2l *are constrained to move, the one along a horizontal line* Ox, *and the other along a vertical line* Oy. *If the whole system turn round* Oy *with a uniform angular velocity* ω, *it is required to find the positions of equilibrium and the time of a small oscillation.*

Let x, y be the co-ordinates of G the middle point of the rod, θ the angle OAB which the rod makes with Ox. Let R, R' be the reactions at A and B resolved in the plane xOy.



The accelerations of any element dr of the rod whose co-ordinates are (ξ, η) are $\frac{d^2\xi}{dt^2} - \omega^2\xi$ parallel to Ox, $\frac{1}{\xi}\frac{d}{dt}(\xi^2\omega)$ perpendicular to the plane xOy and $\frac{d^2\eta}{dt^2}$ parallel to Oy.

As it will not be necessary to take moments about Ox, Oy, or to resolve perpendicular to the plane xOy, the second acceleration will not be required. The resultants of the effective forces $\frac{d^2\xi}{dt^2}dr$ and $\frac{d^2\eta}{dt^2}dr$, taken throughout the body, are $2l\frac{d^2x}{dt^2}$ and $2l\frac{d^2y}{dt^2}$ acting at G, and a couple $2lk^2\frac{d^2\theta}{dt^2}$ tending to turn the body round G. The resultants of the effective forces $\omega^2\xi dr$ taken throughout the body is a single force acting at G

$$=\int_{-l}^{+l}\omega^2(x + r\cos\theta)\,dr = \omega^2 x \,.\, 2l,$$

and a couple* round G

$$=\int_{-l}^{+l}\omega^2(x + r\cos\theta)\,r\sin\theta\,dr = \omega^2 \,.\, 2l \,.\, \frac{l^2}{3}\sin\theta\cos\theta,$$

the distance r being measured from G towards A.

Then we have, by resolving along Ox, Oy, and by taking moments about G, the *dynamical equations*

$$\left.\begin{aligned} 2l\,.\,\frac{d^2x}{dt^2} &= -R' + \omega^2 x\,.\,2l \\ 2l\,.\,\frac{d^2y}{dt^2} &= -R + g\,.\,2l \\ 2l\,.k^2\,.\,\frac{d^2\theta}{dt^2} &= Rx - R'y - \omega^2\,.\,2l\,.\,\frac{l^2}{3}\sin\theta\cos\theta \end{aligned}\right\}\,\cdots\cdots\cdots\ (1).$$

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* If a body in one plane be turning about an axis in its own plane with an angular velocity ω, a general expression can be found for the resultants of the centrifugal forces on all the elements of the body. Take the centre of gravity G as origin and the axis of y parallel to the fixed axis. Let c be the distance

We have also the *geometrical equations*

$$\left.\begin{aligned} x &= l\cos\theta \\ y &= l\sin\theta \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Eliminating R, R', from the equations (1), we get

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} + k^2\frac{d^2\theta}{dt^2} = gx - \omega^2 xy - \omega^2\frac{l^2}{3}\sin\theta\cos\theta \ldots\ldots (3).$$

To find the position of rest.

We observe that if the rod were placed in that position it would always remain there, and that therefore $\frac{d^2x}{dt^2} = 0$, $\frac{d^2y}{dt^2} = 0$, $\frac{d^2\theta}{dt^2} = 0$. This gives

$$gx - \omega^2 xy - \omega^2\frac{l^2}{3}\sin\theta\cos\theta = 0 \ldots\ldots\ldots\ldots (4).$$

Joining this with equations (2), we get $\theta = \frac{\pi}{2}$, or $\sin\theta = \frac{3g}{4\omega^2 l}$, and thus the positions of equilibrium are found. Let any one of these positions be represented by $\theta = \alpha$, $x = a$, $y = b$.

To find the motion of oscillation.

Let $x = a + x'$, $y = b + y'$, $\theta = \alpha + \theta'$, where x', y', θ' are all small quantities, then we must substitute these values in equation (3). On the left-hand side since $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2\theta}{dt^2}$, are all small, we have simply to write a, b, α, for x, y, θ. On the right-hand side the substitution should be made by Taylor's Theorem, thus

$$f(a + x', b + y', \alpha + \theta') = \frac{df}{da}x' + \frac{df}{db}y' + \frac{df}{d\alpha}\theta'.$$

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of G from the axis of rotation. Then all the centrifugal forces are equivalent to a single resultant force at G

$$= \int\omega^2(c + x)\,dm = \omega^2 . Mc, \text{ since } \bar{x} = 0,$$

and to a single resultant couple

$$= \int\omega^2(c + x)\,y\,dm, = \omega^2\int xy\,dm, \text{ since } \bar{y} = 0.$$

We know that the first term $f(a, b, \alpha)$ will be equal to nothing, because this was the very equation (4) from which a, b, α were found. We therefore get

$$a\frac{d^2y'}{dt^2} - b\frac{d^2x'}{dt^2} + k^2\frac{d^2\theta'}{dt^2} = (g - \omega^2 b)\,x' - \omega^2 a y' - \omega^2\frac{l^2}{3}\cos 2\alpha\,.\,\theta'.$$

But by putting $\theta = \alpha + \theta'$ in equations (2), we get by Taylor's Theorem $x' = -l\sin\alpha\,.\,\theta'$, $y' = l\cos\alpha\,.\,\theta'$.

Hence the equation to determine the motion is

$$(l^2 + k^2)\frac{d^2\theta'}{dt^2} + \left(gl\sin\alpha + \frac{4}{3}\omega^2 l^2\cos 2\alpha\right)\theta' = 0.$$

Now, if $gl\sin\alpha + \frac{4}{3}\omega^2 l^2\cos 2\alpha = n$ be positive when either of the two values of α is substituted, that position of equilibrium is *stable*, and the time of a small oscillation is

$$2\pi\sqrt{\frac{l^2 + k^2}{n}}\,.$$

If n be negative the equilibrium is *unstable*, and there can be no oscillation.

If $\omega^2 > \frac{3g}{4l}$ there are two positions of equilibrium of the rod. It will be found by substitution that the position in which the rod is inclined to the vertical is stable, and the other position unstable. If $\omega^2 < \frac{3g}{4l}$ the only position in which the rod can rest is vertical, and this position is stable.

If $n = 0$, the body is in a position of neutral equilibrium. To determine the small oscillations we must retain terms of an order higher than the first. By a known transformation we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = \frac{d}{dt}\left(l^2\frac{d\theta}{dt}\right).$$

Hence the left-hand side of equation (3) becomes $(l^2+k^2)\dfrac{d^2\theta}{dt^2}$.
The right-hand side becomes by Taylor's Theorem

$$\frac{d^2}{d\alpha^2}\left(gl\cos\alpha-\frac{2}{3}\,\omega^2l^2\sin 2\alpha\right)\frac{\theta'^2}{1\,.\,2}+\&c.$$

When $n=0$, we have $\alpha=\dfrac{\pi}{2}$ and $\omega^2=\dfrac{3g}{4l}$. Making the necessary substitutions the equation of motion becomes

$$(l^2+k^2)\frac{d^2\theta'}{dt^2}=-\frac{gl}{2}\,\theta'^3.$$

Since the lowest power of θ' on the right-hand side is odd and its coefficient negative, the equilibrium is stable for a displacement on either side of the position of equilibrium. Let α be the initial value of θ', then the time T of reaching the position of equilibrium is

$$T=\sqrt{\frac{4\,(l^2+k^2)}{gl}}\int_0^\alpha\frac{d\theta'}{\sqrt{\alpha^4-\theta'^4}},$$

put $\theta'=\alpha\phi$, then

$$T=\sqrt{\frac{4\,(l^2+k^2)}{gl}}\,.\int_0^1\frac{d\phi}{\sqrt{1-\phi^4}}\,.\,\frac{1}{\alpha}.$$

Hence the time of reaching the position of equilibrium varies inversely as the arc. When the initial displacement is indefinitely small, the time becomes infinite.

192. This problem might have been easily solved by the first method. For if the two perpendiculars to Ox, Oy at A and B meet in N, N is the instantaneous axis. Taking moments about N, we have the equation

$$(l^2+k^2)\frac{d^2\theta}{dt^2}=gl\cos\theta-\int_{-l}^{+l}\omega^2\,(l+r)^2\sin\theta\cos\theta\,\frac{dr}{2l}$$

$$=gl\cos\theta-\frac{4l^2}{3}\,.\sin\theta\cos\theta$$

$$=f(\theta).$$

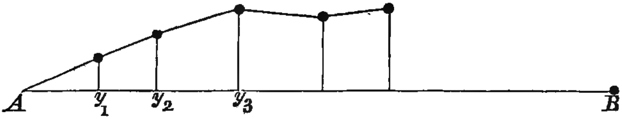
Then the positions of equilibrium can be found from the equation

$$f(\alpha) = 0,$$

and the time of oscillation from the equation

$$(l^2 + k^2)\frac{d^2\theta'}{dt^2} = \frac{df(\alpha)}{d\alpha}.\theta'.$$

193. Ex. 2. *A string of length* (n + 1) l, *and insensible mass, stretched between two fixed points with a force* T, *is loaded at intervals* l *with* n *equal masses* m *not under the influence of gravity and is slightly disturbed; if* $\frac{\mathrm{T}}{\mathrm{lm}} = \mathrm{c}^2$, *prove that the periodic times of the simple transversal vibrations which in general coexist are given by the formula* $\frac{\pi}{\mathrm{c}}$ *cosec* $\left(\frac{\mathrm{i}}{\mathrm{n}+1}\frac{\pi}{2}\right)$ *on putting in succession* i = 1, 2, 3 ... n.



Let A, B be the fixed points; $y_1, y_2 \dots y_n$ the ordinates at time t of the n particles.

The motion of the particles parallel to AB is of the second order, and hence the tensions of all the strings must be equal, and in the small terms we may put this tension equal to T.

Consider the motion of the particle whose ordinate is y_k. The equation of motion is

$$m\frac{d^2y_k}{dt^2} = \frac{y_{k+1} - y_k}{l}T - \frac{y_k - y_{k-1}}{l}T;$$

$$\therefore \frac{d^2y_k}{dt^2} = c^2(y_{k+1} - 2y_k + y_{k-1}).$$

Now the motion of each particle is vibratory, we may therefore expand y_k in a series of the form

$$y_k = \Sigma L \sin (pt + \alpha).$$

As there may be a term of the argument pt in every y, let $L_1, L_2, \dots$ be their respective coefficients. Then substituting, we have

$$L_{k+1} - 2L_k + L_{k-1} = -\frac{p^2}{c^2} L .$$

To solve this, put $L_k = a^k L$, we get

$$a - 2 + \frac{1}{a} = -\left(\frac{p}{c}\right)^2,$$

or
$$\left.\begin{aligned} \sqrt{a} - \frac{1}{\sqrt{a}} &= \frac{p}{c}\sqrt{-1} \\ \sqrt{a} + \frac{1}{\sqrt{a}} &= \pm 2\sqrt{1 - \left(\frac{p}{2c}\right)^2} \end{aligned}\right\};$$
and

$$\therefore \sqrt{a} = \pm\sqrt{1 - \left(\frac{p}{2c}\right)^2} + \frac{p}{2c}\sqrt{-1}.$$

Let these roots be called a_1 and a_2, then we have

$$L_k = Aa_1^k + Ba_2^k;$$

$$\therefore y_k = \Sigma \left[Aa_1^k + Ba_2^k\right] \sin (pt + \alpha).$$

Now $y = 0$ when $k = 0$ for *all* values of t, hence every term of the series must vanish;

$$\therefore A + B = 0.$$

Also $y = 0$ when $k = n + 1$ for all values of t;

$$\therefore Aa_1^{n+1} + Ba_2^{n+1} = 0.$$

These equations give by division $a_1^{n+1} = a_2^{n+1}$. But if $\frac{p}{2c} > 1$, the ratio of a_1 to a_2 is real and different from unity. Hence we must have $\frac{p}{2c} < 1$. Let then $\frac{p}{2c} = \sin\theta$;

$$\therefore a = \cos 2\theta \pm \sin 2\theta \sqrt{-1}.$$

Hence, by what we proved before,

$$(\cos 2\theta + \sin 2\theta \sqrt{-1})^{n+1} = (\cos 2\theta - \sin 2\theta \sqrt{-1})^{n+1};$$

$$\therefore \sin 2(n+1)\theta = 0;$$

$$\therefore 2(n+1)\theta = i\pi;$$

$$\therefore \theta = \frac{i}{n+1} \cdot \frac{\pi}{2};$$

$$\therefore \frac{p}{2c} = \sin \frac{i}{n+1} \cdot \frac{\pi}{2},$$

and the period of any term $= \dfrac{2\pi}{p}$.

If m and l be indefinitely small and n indefinitely large, the loaded string may be regarded as a uniform string of length $(n+1)l = L$ and mass $nm = M$ stretched between two fixed points with a tension T. In this case the expression just found reduces to $p = \pi i \sqrt{\dfrac{T}{ML}}$.

Third Method.

194. When the system of bodies in motion admits of only one independent motion, the time of a small oscillation may frequently be deduced from the equation of Vis Viva. This equation will be one of the second order of small quantities, and in forming the equation it will be necessary to take into account small quantities of that order. This will sometimes involve rather troublesome considerations. On the other hand the equation will be free from all the unknown reactions, and we may thus frequently save much elimination.

The method of proceeding will be made clear by the following example, by which a comparison may be made with the method of Art. 189.

The motion of a body in space of two dimensions is given by the co-ordinates x, y *of its centre of gravity, and* *the angle θ which any fixed line in the body makes with a line fixed in space. The body being in equilibrium under the action of gravity it is required to find the time of a small oscillation.*

Since the body is capable of only one independent motion, we may express (x, y) as functions of θ, thus

$$x = F(\theta), \quad y = f(\theta).$$

Let Mk^2 be the moment of inertia of the body about its centre of gravity, then the equation of Vis Viva becomes

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + k^2\left(\frac{d\theta}{dt}\right)^2 = 2g\,(y_0 - y),$$

where y_0 is the altitude of the centre of gravity in the position of equilibrium.

Let α be the value of θ when the body is in the position of equilibrium, and suppose that at the time t, $\theta = \alpha + \phi$. Then, by M'Laurin's theorem,

$$y = y_0 + y_0'\phi + y_0''\frac{\phi^2}{2} + \ldots,$$

where y_0', y_0'' are the values of $\frac{dy}{d\theta}, \frac{d^2y}{d\theta^2}$ when $\theta = \alpha$. But in the position of equilibrium y is a maximum or minimum; $\therefore\ y_0' = 0$. Hence the equation of Vis Viva becomes

$$(x_0'^2 + k^2)\left(\frac{d\phi}{dt}\right)^2 = -gy_0''\phi^2,$$

where x_0' is the value of $\frac{dx}{d\theta}$ when $\theta = \alpha$; differentiating we get

$$(x_0'^2 + k^2)\frac{d^2\phi}{dt^2} = -gy_0''\phi.$$

If L be the length of the simple equivalent pendulum, we have

$$L = \frac{k^2 + \left(\frac{dx}{d\theta}\right)^2}{\frac{d^2y}{d\theta^2}}.$$

where for θ we are to write its value α after the differentiations have been effected. It is not difficult to see that the geometrical meaning of this result is the same as that given in Art. 189. This formula is given by Mr Holditch in the *Camb. Trans.* Vol. VIII.

In just the same way we may deduce a formula for the length of the equivalent pendulum in the case of a body moving in space of three dimensions, see Art. 166.

195. When the system of bodies admits of several independent displacements, the principle of Vis Viva which furnishes *one* equation will be insufficient for the solution of the problem. In such a case we may have recourse to the general equations of Lagrange given in Art. 165.

As an example of this method, let us consider a problem connected with the motion of the balls in Watt's Governor of the steam engine. The mode in which this works to moderate the fluctuations of the engine is well known. A somewhat similar apparatus has been used to regulate the motion of clocks, and in other cases where uniformity of motion is required. If there be any increase in the driving power of the engine, or any diminution of the load, so that the engine begins to move too fast, the balls, by their increased centrifugal force, open outwards, and by means of a lever either cut off the driving power or increase the load by a quantity proportional to the angle opened out. If on the other hand the engine goes too slow, the balls fall inward, and more driving power is called into action. In the case of the steam engine the lever is attached to the throttle-valve, and thus regulates the supply of steam. It is clear that a complete adaptation of the driving power to the load cannot take place instantaneously, but the machine will make a series of small oscillations about a mean state of steady motion. The problem to be considered may therefore be stated thus:—

Two equal rods OA, OA', each of length l, are connected with a vertical spindle by means of a hinge at O which permits free motion in the vertical plane AOA'. At A and A' are attached two balls, each of mass m. To represent the inertia of the other parts of the engine we shall suppose a horizontal fly-wheel attached to the spindle, whose moment of inertia about the spindle is I. When the machine is in uniform motion, the rods are inclined at some angle α to the vertical, and turn round it with uniform angular velocity n. If, owing to any disturbance of the motion, the rods have opened out to an angle θ with the vertical, a force is called into play whose moment about the spindle is $-\beta(\theta-\alpha)$. It is required to find the oscillations about the state of steady motion.

Let ϕ be the angle the plane AOA' makes with some vertical plane fixed in space. The equation of angular momentum about the spindle is

$$\frac{d}{dt}(I+2mk^2\sin^2\theta)\frac{d\phi}{dt}=-\beta(\theta-\alpha)\,\ldots\ldots\ldots(1),$$

where mk^2 is the moment of inertia of a rod and ball about a perpendicular to the rod through O, the balls being regarded as indefinitely small heavy particles.

The semi Vis Viva of the system is

$$T=\tfrac{1}{2}I\left(\frac{d\phi}{dt}\right)^2+mk^2\left\{\left(\frac{d\theta}{dt}\right)^2+\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right\},$$

and the moment of the impressed forces on either rod and ball about a horizontal through O perpendicular to the plane AOA' is $\frac{1}{2}\frac{dU}{d\theta}=-gh\sin\theta$, where h is the distance of the centre of gravity of a rod and ball from O. Hence by Lagrange's equation

$$\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}=\frac{dU}{d\theta},$$

we have

$$\frac{d^2\theta}{dt^2}-\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2=-\frac{g}{a}\sin\theta\,\ldots\ldots\ldots\ldots\ldots(2),$$

where a has been written for $\frac{k^2}{h}$. This equation might also have been obtained by taking the acceleration of either ball, treated as a particle, in a direction perpendicular to the rod in the plane in which θ is measured.

To find the steady motion we put $\theta = \alpha$, $\dfrac{d\phi}{dt} = n$, the second equation then gives $n^2 \cos\alpha = \dfrac{g}{a}$.

To find the oscillations, we put $\theta = \alpha + x$, $\dfrac{d\phi}{dt} = n + y$. The two equations then become

$$\left.\begin{aligned} (I + 2mk^2 \sin^2\alpha)\frac{dy}{dt} + 2mk^2 n \sin 2\alpha \frac{dx}{dt} &= -\beta x \\ \frac{d^2x}{dt^2} - \sin 2\alpha\, ny = \left(n^2 \cos 2\alpha - \frac{g}{a}\cos\alpha\right) x & \end{aligned}\right\}.$$

To solve these equations, we must write them in the form

$$\left.\begin{aligned} \left(\sin 2\alpha\, D + \frac{\beta}{2mk^2 n}\right) nx + \left(\frac{I}{2mk^2} + \sin^2\alpha\right) Dy &= 0 \\ (D^2 + n^2 \sin^2\alpha)\, x - n \sin 2\alpha\, y &= 0 \end{aligned}\right\},$$

where the symbol D stands for the operation $\dfrac{d}{dt}$. Eliminating y by cross multiplication we have

$$\left[\left(\frac{I}{2mk^2} + \sin^2\alpha\right) D^3 + n^2 \sin^2\alpha \left(1 + 3\cos^2\alpha + \frac{I}{2mk^2}\right) D + \frac{\beta}{2mk^2}\, n \sin 2\alpha\right] x = 0.$$

The real root of this cubic equation is necessarily negative because the last term is positive. The other two roots are imaginary because the term D^2 has disappeared between two terms of like signs. Also the sum of the three roots being zero, the real parts of the two imaginary roots must be positive. Let these roots therefore be $-r$ and $p \pm q\sqrt{-1}$. Then

$$x = He^{-rt} + Ke^{pt} \sin(qt + L),$$

where H, K, L are three undetermined constants depending on the nature of the initial disturbance. Thus it appears that the oscillation is unstable*. The balls will alternately approach and recede from the vertical spindle with increasing violence.

196. When the motion of clock-work is regulated by centrifugal balls, it is found as a matter of observation that there is a strong tendency to irregularity. If the balls once receive in the slightest degree an elliptic motion, the resistance $\beta(\theta-\alpha)$ by which the motion of the balls is regulated may tend to render the ellipse more and more elliptical. To correct this some other resistance must be called into play. This resistance should be of such a character that it does not affect the circular motion and is only produced by the ellipticity of the movement. In an equatoreal clock at Liverpool, the elliptic motion of the balls causes a slider on the vertical spindle to rise and fall. This is connected by a lever with a plate of metal moving in a vessel of water. The resistance of the water quickly reduces the elliptic to circular motion and has no effect on the circular motion. See *Astronomical Notices*, XI. page 17, 1851.

The general effect of the water will be to produce a resistance varying as the velocity, and may therefore be represented by a term $-\gamma\dfrac{d\theta}{dt}$ on the right hand of equation (2). The solution being continued as before, the cubic will now take the form

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* That the general effect of the force βx is to make the motion unstable may be shown thus. Let the balls be moving in steady motion with a given angular momentum equal to A. Then we have $(I+2mk^2\sin^2\alpha)n=A$, but since $an^2\cos\alpha=g$, we get $\dfrac{(I+2mk^2\sin^2\alpha)^2}{a\cos\alpha}=\dfrac{A^2}{g}$. Now let us suppose the rods slightly displaced so as to make an angle θ, greater than α, with the vertical, and to be making small oscillations about their mean position. Let the force $-\beta(\theta-\alpha)$, thus called into play, be supposed to act as an impulsive force at small intervals of time. Consider first the effect of any one impulse. Its effect will be to diminish A, the angular momentum. It follows from the above equation that α is also diminished. So that the angular distance $\theta-\alpha$ of the ball from its mean position is increased, that is, the arc of oscillation is increased. In the same way it may be shown that when θ is less than α, the effect of an impulse is to increase A and α, and thus again to increase the arc of oscillation. Thus the arc being always increased, the whole effect is to make the motion unstable.

$$\left[\left(\frac{I}{2mk^2}+\sin^2\alpha\right)(D^3+\gamma D^2)+n\sin^2\alpha\left(1+3\cos^2\alpha+\frac{I}{2mk^2}\right)D\right.$$
$$\left.+\frac{\beta}{2mk^2}n\sin 2\alpha\right]x=0.$$

If the roots of this cubic are real, they are all negative, and the value of x takes the form

$$x=Ae^{-\lambda t}+Be^{-\mu t}+Ce^{-\nu t},$$

where $-\lambda$, $-\mu$, $-\nu$ are the roots and A, B, C are three undetermined constants. If one root only is real, that root is negative, and if the other two be $p\pm q\sqrt{-1}$ the value of x takes the form

$$x=He^{-rt}+Ke^{pt}\sin(qt+L),$$

where H, K, L as before are undetermined constants.

In order that the motion may be stable it is necessary that p should be negative. The analytical condition* of this is

$$\gamma\left(1+3\cos^2\alpha+\frac{I}{2mk^2}\right)>\frac{\beta}{2mk^2}2\cot\alpha.$$

If γ be sufficiently great this condition may be satisfied. The uniformity of motion of the rods round the vertical will then be disturbed by an oscillation whose magnitude is continually decreasing and whose period is $\frac{2\pi}{q}$. By properly choosing the magnitude of I when constructing the instrument, the period may sometimes be so arranged as to produce the least possible ill effect. If the period be made very long the instrument will work smoothly. If it can be made very short there will be less deviation from circular motion.

In this investigation no notice has been taken of the frictions at the hinge and at the mechanical appliances of the

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* If the roots of the cubic $ax^3+bx^2+cx+d=0$ be $x=\alpha\pm\beta\sqrt{(-1)}$ and γ, we have $-\frac{b}{a}=2\alpha+\gamma$, $\frac{c}{a}=2\gamma\alpha+\alpha^2+\beta^2$, $-\frac{d}{a}=(\alpha^2+\beta^2)\gamma$, whence we easily deduce $\frac{bc-ad}{a^2}=-2\alpha\{(\alpha+\gamma)^2+\beta^2\}$; hence $bc-ad$ and α have always opposite signs.

Governor, which may not be inconsiderable. These in many cases tend to reduce the oscillation and keep it within bounds.

In the case of Watt's Governor if any permanent change be made in the relation between the driving power and the load, the state of uniform motion which the engine will finally assume is different from that which it had before the change. For this reason it has been suggested that the term "Governor" is inappropriate, the instrument being in fact only a "Moderator" of the fluctuations of the engine. The reader who may be interested in the construction of Governors, properly so called, may refer to an article by Mr Siemens in *Phil. Trans.* for 1866. He may also consult a brief sketch by Prof. Maxwell in the *Phil. Mag.* for May, 1868.

197. When the oscillations of the system take place about a position of equilibrium Lagrange has indicated a general method of proceeding by which the result may be shortly arrived at.

PROP. *To determine the oscillations of a system of bodies about a position of equilibrium.*

Let $2T$ be the vis viva of the whole system, which may be found by Art. 166, and let U be the force function. Let θ, ϕ, ... be n certain small independent quantities by which the position of the system may be determined, and whose values are zero when the system is in the position of equilibrium. Also, as before, let accents denote differential coefficients with respect to t. Then we have the n equations

$$\left.\begin{aligned} \frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} &= \frac{dU}{d\theta} \\ \&c. &= \&c. \end{aligned}\right\} \quad \text{.................. (1).}$$

Let x, y, z be the co-ordinates of a particle m, then

$$\left.\begin{aligned} 2T &= \Sigma m\,(x'^2 + y'^2 + z'^2) \\ U &= \Sigma m f\,(x, y, z) \end{aligned}\right\} \text{.............. (2).}$$

The co-ordinates x, y, z are functions of θ, ϕ, ... and may be expanded by Taylor's Theorem in powers of θ, ϕ .... Assuming that the coefficients of the first powers do not all vanish, we may, since θ, ϕ, ... are small quantities, neglect the squares and higher powers. Let then these expansions be

$$x = \alpha + \alpha_1\theta + \alpha_2\phi + \dots\dots\dots\dots\dots\dots (3),$$

with similar expressions for y and z. Here α, α_1, α_2 are constant quantities. Hence differentiating we have

$$x' = \alpha_1\theta' + \alpha_2\phi' + \dots \quad \dots\dots\dots\dots\dots\dots (4),$$

with similar expressions for y' and z'.

Substituting in (2), we see that T is a function of θ', ϕ'... and not of θ, ϕ, ... Hence $\dfrac{dT}{d\theta} = 0$, $\dfrac{dT}{d\phi} = 0$, &c. $= 0$, and the equations (1) reduce to

$$\left.\begin{aligned} \frac{d}{dt}\frac{dT}{d\theta'} &= \frac{dU}{d\theta} \\ \&c. &= \&c. \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (5).$$

Let the values of T and U after substituting from (3) and (4) in the equations (2) be

$$\left.\begin{aligned} T &= A_1\theta'^2 + A_2\phi'^2 + B_1\phi'\psi' \dots \\ U - U_0 &= a_1\theta^2 + a_2\phi^2 + b_1\phi\psi + \dots \end{aligned}\right\} \dots\dots\dots (6),$$

where U_0 is the value of U when the system is in the position of equilibrium, and A_1, a_1, &c. are constants. In these expressions all terms above the second order are to be neglected. The terms in the first order in $U - U_0$ are absent, because by the principle of virtual velocities

$$\delta U = \Sigma m\,(X\delta x + Y\delta y + Z\delta z)$$

vanishes in a position of equilibrium so far as quantities of the first order are concerned. Substituting in equations (5), it is obvious that the n resulting equations will be linear, and of the form

$$\left.\begin{aligned} E_1\frac{d^2\theta}{dt^2} + E_2\frac{d^2\phi}{dt^2} + \dots &= e_1\theta + e_2\phi + \dots \\ F_1\frac{d^2\theta}{dt^2} + F_2\frac{d^2\phi}{dt^2} + \dots &= f_1\theta + f_2\phi + \dots \\ \&c. &= \&c. \end{aligned}\right\} \dots\dots (7),$$

where E_1, e_1, &c. F_1, f_1, &c. are all constants.

To solve these, we must, as usual, assume

$$\theta = k \sin(\lambda t + \kappa),\quad \phi = k' \sin(\lambda t + \kappa),\ \&c.\ \ldots\ldots \ (8).$$

Substituting in the n equations (7) we shall obtain $n-1$ equations to determine the $n-1$ ratios $\frac{k'}{k}$, $\frac{k''}{k}$, &c., together with an equation of the n^{th} degree to determine the n values of λ^2. This substitution may be effected once for all, as follows. The equations (7) become

$$\left.\begin{aligned} -\lambda^2(E_1k + F_2k' + \ldots) &= e_1k + e_2k' + \ldots \\ -\lambda^2(F_1k + F_3k' + \ldots) &= f_1k + f_2k' + \ldots \\ \&c. &= \&c. \end{aligned}\right\}\ \ldots\ldots \ (9).$$

Referring to the manner in which the equations (7) were obtained from (5) and (6), we see that

$$\left.\begin{aligned} \frac{dT}{d\theta'} &= E_1\theta' + E_2\phi' + \ldots \\ \frac{dU}{d\theta} &= e_1\theta + e_2\phi + \ldots \end{aligned}\right\}.$$

Hence, if we put

$$\left.\begin{aligned} T_1 &= A_1k^2 + A_2k'^2 + B_1k'k'' + \ldots \\ U_1 &= a_1k^2 + a_2k'^2 + b_1k'k'' + \ldots \end{aligned}\right\}\ \ldots\ldots\ldots \ (10),$$

which are obtained from the expressions for T and $U - U_0$, in equations (6) by writing k for θ' and θ, k' for ϕ' and ϕ, then

$$\left.\begin{aligned} \frac{dT_1}{dk} &= E_1k + E_2k' + \ldots \\ \frac{dU_1}{dk} &= e_1k + e_2k' + \ldots \end{aligned}\right\}.$$

Hence the equations (7) become

$$\left.\begin{aligned} \lambda^2\frac{dT_1}{dk} + \frac{dU_1}{dk} &= 0 \\ \lambda^2\frac{dT_1}{dk'} + \frac{dU_1}{dk'} &= 0 \\ \&c. &= 0 \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots \ (11).$$

In these equations the quantities k, k', &c. enter in the first power only, and the ratios $k : k' : k''$: &c. may be eliminated by the usual general methods.

In this way we have an equation to determine λ^2, and thus by substituting in (8) the values of θ, ϕ ... may be found.

In any particular problem the process of applying this proposition will be as follows. Write down the equation of Vis Viva in terms of such independent quantities θ, ϕ, &c. that their values in the position of equilibrium are zero. Replace θ^2, ϕ^2, &c. by k^2, k'^2, &c. and θ'^2, ϕ'^2, &c. by $-\lambda^2 k^2$, $-\lambda'^2 k'^2$, &c. and differentiate the equation partially with regard to each of the n small independent quantities k, k', &c.... and we shall thus obtain n equations which are identical with those written down in (11).

198. PROP. *To explain the principle of the co-existence of small oscillations.*

It has been proved that the motion of any system is made up of a number of simultaneous oscillations whose general type is

$$k \sin (\lambda t + \kappa).$$

Each of these motions is called *a simple oscillation*, and if the initial conditions be properly chosen, any one term will give the law of motion, and the system will make small oscillations analogous to those of a simple pendulum. Thus the general motion of a system of bodies is made up of all the simple oscillations of which it is capable. The number of such simple oscillations is not necessarily equal to the number of moveable bodies, but is equal to the number of independent motions. When the periods of all these simple oscillations are commensurable, the whole system will return to the same state in a period equal to the least common multiple of these periods.

199. PROP. *If a system of bodies be in equilibrium under the action of any forces, to determine whether the equilibrium is stable or unstable*.*

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* See Lagrange's *Mécanique Analytique;* Duhamel's *Cours de Mécanique*, Vol. II.; and Vieille's *Cours Complémentaire d'Analyse et de Mécanique Rationelle.*

Let the system receive any small disturbance and let the type of the resulting motion be $k \sin(\lambda t+\kappa)$. Then, following the same notation as before, the equation to determine λ is found by eliminating the ratios $k : k' :: k'' :$ &c. from the equations

$$\left.\begin{aligned} \lambda^2\frac{dT_1}{dk}+\frac{dU_1}{dk}&=0 \\ \lambda^2\frac{dT_1}{dk'}+\frac{dU_1}{dk'}&=0 \\ \&\text{c.}&=0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

There are three different forms which the type of motion may assume, according to the nature of the values of λ.

I. If all the values of λ^2 be positive, the type of motion is $k\sin(\lambda t+\kappa)$. In this case the motion is represented by a number of simultaneous small oscillations about the position of equilibrium. The system never departs far from the position of equilibrium, and the equilibrium is said to be *stable.*

II. If any value of λ^2 be negative, the corresponding trigonometrical expression takes the form $Ae^{at}+Be^{-at}$.

In this case the motion is said to be *unstable,* for θ, ϕ, &c. will generally become large if a sufficient time t be allowed to elapse. It is of course possible that the initial conditions may be such that A vanishes, and in this case the system will not depart far from the position of equilibrium. The equilibrium in such a case may be said to be stable for some displacements and unstable for others. It is however more usual to call a position of equilibrium unstable, if any small displacements can be found such that the system will not then remain in the immediate neighbourhood of the position of equilibrium.

III. Lastly, let any value of λ^2 be imaginary, thus

$$\lambda^2=p\pm q\sqrt{-1}, \text{ then } \lambda=\pm(\alpha\pm\beta\sqrt{-1}).$$

The corresponding trigonometrical expression takes the form

$$(Ae^{\alpha t}+Be^{-\alpha t})\sin(\beta t+\gamma).$$

In this case the motion is oscillatory, but as the magnitude of the oscillations continually increases with the time, the equilibrium is said to be unstable.

When a system is in equilibrium we know from Statics that $\Sigma m (Xdx + Ydy + Zdz) = 0$, so that in general the function U is either a minimum or a maximum. *First,* let it be a minimum, then we may show that the equilibrium is unstable. For if possible let it be stable, then the motion resulting from any disturbance is represented by the type $k \sin (\lambda t + \kappa)$, where k is essentially real.

Now multiplying the equations (1) respectively by k, k', &c. and adding we get, since T_1 and U_1 are homogeneous functions of the second order,

$$\lambda^2 T_1 + U_1 = 0,$$

$$\therefore \lambda^2 = -\frac{U_1}{T_1} \qquad\qquad (2).$$

Now U is a minimum for all small variations of θ, ϕ, &c., hence $U - U_0$ is positive for all values of θ, ϕ, &c. less than certain fixed limits, which are supposed not to be very small. Also U_1 is obtained from $U - U_0$ by writing for θ, ϕ, &c. the very small real quantities k, k' &c. Hence U_1 is also positive. Again, since $2T$ is the Vis Viva of the system, it is a function of θ', ϕ', &c. which is essentially positive for all values of θ', ϕ', &c. Hence T_1 which is obtained from T by writing k, k', &c. for θ', ϕ', &c. is also positive. Hence by equation (2) the values of λ^2 are all negative. Hence there is no real term of the form $k \sin (\lambda t + \kappa)$. That is, *the equilibrium is unstable.*

Secondly, let $\Sigma m (Xdx + Ydy + Zdz)$ be a maximum, then we may more conveniently show that the equilibrium is stable in the following manner. Let the values of θ, ϕ, &c. in the position of equilibrium be α, β, &c. Suppose the system displaced into any position defined by $\theta = \alpha + p_0$, $\phi = \beta + q_0$, &c. very near the position of equilibrium, so that p_0, q_0, &c. are all very small, and suppose the system to be set in motion with any small velocity. Let V_0 be the velocity thus communicated to the particle m. At any time t after the commencement of the motion, let the values of the variables θ, ϕ, &c. be $\alpha+p$, $\beta+q$, &c., and let v be the velocity of the particle m. Also let us represent the force function by $f(\theta, \phi \ldots)$. Then assuming that the geometrical equations do not contain the time explicitly, we have, by Vis Viva,

$$\frac{1}{2}\Sigma mv^2 - \frac{1}{2}\Sigma mv_0^2 = f(\alpha+p, \text{ \&c.}) - f(\alpha, \text{ \&c.}).$$

We shall now show that if the variables p, q, &c. do not always remain very small, a time must come when Σmv^2 is negative, which is clearly impossible.

Since the function f is a maximum, we shall have

$$f(\alpha, \beta, \ldots) - f(\alpha+p, \beta+q, \ldots),$$

positive for all values of p, q, &c. numerically less than certain fixed values which we shall represent by p', q' ..., and which may be as small as we please. Let them be taken sufficiently great that the initial values p_0, q_0, &c. are all respectively less than p', q', &c. If we represent this positive function by $F(p, q, \ldots)$ the equation of Vis Viva may be written

$$\frac{1}{2}\Sigma mv^2 - \frac{1}{2}\Sigma mv_0^2 = F(p_0, q_0, \ldots) - F(p, q, \ldots).$$

Now $F(0, 0, \ldots) = 0$, and as p, q, &c. increase or decrease, $F(p, q, \ldots)$ remains positive for all values of p, q, &c. numerically less than p', q', &c. Let A be the *least* of the values of $F(p, q, \ldots)$ when we give to one or more of the variables p, q, &c. their limiting values. Then A is a positive quantity greater than zero. Let us also suppose the system to be started from its initial position $\alpha+p_0$, $\beta+q_0$, with so small an initial Vis Viva, that

$$\frac{1}{2}\Sigma mv_0^2 + F(p_0, q_0 \ldots) < A,$$

then throughout the motion we must have

$$A - F(p, q, \ldots) = \text{a positive quantity.}$$

If then $p, q, \ldots$ do not always remain less than p', q',..., there must be a time when one, at least, of them as they increase numerically, becomes equal to its limit, and at that instant the function F is equal or greater than A. So that Σmv^2 has a negative value, which is impossible. Hence *the equilibrium is stable.*

Oscillations of the Second Order.

200. When the oscillations of a system are very small it will generally be a sufficient approximation to neglect the squares of the small quantities which determine the change of position of the system. But there are cases in which it is necessary to proceed to a second approximation. In such cases we may either solve the equations of motion by the method of Variation of Parameters, or we may proceed by the more usual method of approximation which is employed in the Lunar Theory. We shall here consider the latter of these two methods.

In order to make the explanation clearer let us suppose that the system admits of but one independent motion, so that its position may be made to depend on one single small quantity x. Suppose the equation of motion to be

$$\frac{d^2x}{dt^2}+a\frac{dx}{dt}+bx=c+\phi\left(x,\frac{dx}{dt},\ \frac{d^2x}{dt^2}\ldots\right)\ldots\ldots\ldots(1),$$

in which all the quantities of the first order have been collected together on the left-hand side, and those of the second and higher orders form collectively the function ϕ.

As a first approximation we neglect the small function ϕ, and integrating the equation, we have

$$x=\frac{c}{b}+e^{-\frac{at}{2}}\left(A\sin\sqrt{b-\frac{a^2}{4}}\,t+B\cos\sqrt{b-\frac{a^2}{4}}\,t\right)\ldots.(2).$$

To obtain a second approximation we substitute this value of x in the terms on the right-hand side of the equation.

In order to facilitate the integration of the new differential equation thus formed, it will be necessary to expand the powers of the sines and cosines which may occur in the substitution in a series of sines and cosines of multiple angles. This may be effected by the formulæ given in books on Trigonometry. The equation will then take the form

$$\frac{d^2x}{dt^2}+a\frac{dx}{dt}+bx=\Sigma F.e^{pt}\sin(qt+f)\ldots\ldots\ldots(3),$$

where F, p, q, f are quantities given by the expansion, and p must be negative if the oscillation is to remain small.

Each term on the right side of the equation gives a corresponding term in the expansion of x. To find the term corresponding to the one written down we proceed thus. Let the operation $\dfrac{d}{dx}$ be written D, then

$$x = \frac{1}{D^2 + aD + b}\, Fe^{pt} \sin (qt + f)$$

$$= Fe^{pt} \frac{1}{(D+p)^2 + a(D+p) + b} \sin (qt + f)$$

by the known theorem

$$f(D)\, e^{pt} f(t) = e^{pt} f(D+p)\, f(t),$$

in the Differential Calculus. Putting

$$a' = 2p + a \text{ and } b' = p^2 + ap + b,$$

this may be written

$$x = F \,.\, e^{pt} \,.\, \frac{1}{D^2 + a'D + b'} \,.\, \sin (qt + f)$$

$$= F \,.\, e^{pt} \,.\, \frac{D^2 + b' - a'D}{(D^2 + b')^2 - a'^2 D^2} \sin (qt + f).$$

Now by another theorem in the Differential Calculus

$$\psi (D^2) \sin qt = \psi (-q^2) \sin qt,$$

$$\therefore\ x = F \,.\, e^{pt} \,.\, \frac{(-q^2 + b') \sin (qt + f) - a'q \cos (qt + f)}{(-q^2 + b')^2 + a'^2 q^2},$$

which may be written for brevity

$$x = e^{pt} \{L \sin (qt + f) + M \cos (qt + f)\}.$$

Thus the complete value of x is

$$x = \frac{c}{b} + e^{-\frac{at}{2}} \left\{A \sin \sqrt{b - \frac{a^2}{4}}\, t + B \cos \sqrt{b - \frac{a^2}{4}}\, t\right\}$$

$$+ \Sigma e^{pt} \{L \sin (qt + f) + M \cos (qt + f)\} \ \ldots\ldots\ldots \ (4).$$

In this expression A and B are two arbitrary constants; and, if determined from the initial conditions, it is obvious that their values as determined by using the first or second approximate values of x may be different. This difference will be small, being of the order of the terms L, M. But as it cannot be neglected, it will be necessary either to apply corrections to the values of A, B determined from the first approximate value of x, or to keep A and B undetermined until the form of the complete value of x has been found.

201. There are two cases in the preceding investigation which merit particular observation.

First. If a body be acted on by several forces, each tending to produce oscillatory motions, the effects of these forces are not necessarily proportional to their respective magnitudes. Each separate force will produce terms of the form represented on the right-hand side of the differential equations (3). The force may be such that the coefficient F may be small, yet if the denominators of the expressions for L, M be small, this force may produce a very large effect on the value of x. Thus in forming our differential equation correct to the second order we should include in it all terms of the third or even higher powers of x, which are such that the coefficients of t in the exponent (viz. p) and in the arc (viz. q) are respectively nearly equal to $-\frac{a}{2}$ and $\sqrt{b-\frac{a^2}{4}}$. The process of approximation is therefore somewhat tentative. If at any stage of our approximation a term is introduced much larger than those compared with which it had been already neglected, we may have to revise our whole process and include these additional terms in our approximation at an earlier stage.

Secondly. It is possible that a term in the differential equation (3) may be such that on solving it in the manner above indicated we find $a' = 0$, $b' = q^2$. The value of x is then infinite. This indicates that the expression for x changes its character. Following the rules given in books on Differential Equations, we find

$$x = e^{pt} . t (L \sin qt + M \cos qt) \text{ .... } (5),$$

thus the value of x containing a factor proportional to the time may become very large. This will generally be found to occur when the body is acted on by a small force whose period is so timed that either it always urges the body in the direction in which, according to the first approximation, it is then moving, or always in the opposite direction. The motion thus produced continually accumulates, and our first approximation soon ceases to give a correct representation of the motion.

The effect of this small force may ultimately be to destroy all small oscillations whatever, but this is not necessarily the case. It may only so alter the character of the motion that our assumed first approximation is not sufficiently near the truth to serve as a basis for a series of approximations. We must therefore amend our first approximation by including in it the effect of this small force.

Suppose that in forming our differential equation (3) we find a term of the form

$$e^{-\frac{at}{2}}(F\sin\beta t + G\cos\beta t) \;\ldots\ldots\ldots\ldots\ldots\ldots\ (6),$$

where $\beta = \sqrt{b - \frac{a^2}{4}}$. In this case we find L and M both infinite. Let us then alter our first approximate value of x given in (2) into

$$x = \frac{c}{b} + e^{rt}\{A\sin st + B\cos st\} \;\ldots\ldots\ldots\ldots\ (7).$$

Then repeating our process, beginning with this value of x, we *may* find as our differential equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = c + e^{rt}(F\sin st + G\cos st) \;\ldots\ldots\ (8),$$

where F and G are known functions of A and B. Here the terms on the right-hand side are those which would become (6) if we put $r = -\frac{a}{2}$, $s = \beta$. Then r and s are, if possible, to be so chosen that the value of x given in (7) may satisfy this differential equation.

Substituting and equating the coefficients of $\sin st$ and $\cos st$, we get

$$\left.\begin{array}{l} A\,(r^2+ar+b-s^2)-B\,(2r+a)\,s=F \\ A\,(2r+a)\,s+B\,(r^2+ar+b-s^2)=G \end{array}\right\}.$$

In these equations A and B are to be left arbitrary, being dependent on the initial conditions, while r and s are at our disposal. Now F and G are small quantities of the second order at least. If they were zero, we should have by hypothesis $r=-\frac{a}{2}$ and $s=\beta$, where $\beta=\sqrt{b-\frac{a^2}{4}}$. Let us then put $r=-\frac{a}{2}+\rho$, $s=\beta+\sigma$.

In many cases it will be sufficient to neglect the squares of ρ and σ: substituting we get

$$\left.\begin{array}{l} \rho=-\dfrac{1}{2\beta}\cdot\dfrac{FB-GA}{A^2+B^2} \\ \sigma=-\dfrac{1}{2\beta}\cdot\dfrac{FA+GB}{A^2+B^2} \end{array}\right\}.$$

Introducing these values of r and s in equation (7), we have obtained the proper form of our first approximation.

It is to be remarked that ρ does not necessarily vanish when a vanishes. So that even if the right-hand side of the equation does not present any exponential, it may be necessary to introduce one into the value of x. F, G are known functions of A and B given by the process of substitution. If $\frac{F}{G}=\frac{A}{B}$ it will be unnecessary to introduce any exponential into the value of x. So also if $\frac{F}{G}=-\frac{B}{A}$, it will be unnecessary to alter the coefficient of t.

On substituting the amended first approximate value of x given in (7) in the differential equation, it may happen that the equation does not take the form given in (8); so that it is not possible to satisfy the equation with the value of x assumed in (7). This indicates that some other form must be assumed as a first approximation. What this should be must depend on the particular problem under consideration. The criteria that the right form for a first approximation has been chosen are, that it satisfies the equation to the first order, and that at no subsequent stage of our process of approximation does any term make its appearance which can become so great as to make our assumed value for x cease to be a first approximation.

202. Ex. *A pendulum swings in a very rare medium, resisting partly as the velocity and partly as the square of the velocity, to find the motion.*

Let θ be the angle the straight line joining the point O of support to the centre of gravity G of the pendulum makes with the vertical. Then the equation of motion is

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=-2\kappa\frac{d\theta}{dt}-\mu\left(\frac{d\theta}{dt}\right)^2,$$

where l is the length of the simple equivalent pendulum, 2κ and μ the coefficients of the resistance divided by the moment of inertia of the pendulum about the axis of suspension. Let $\frac{g}{l}=n^2$.

Since θ is small we may write the equation in the form

$$\frac{d^2\theta}{dt^2}+n^2\theta=-2\kappa\frac{d\theta}{dt}-\mu\left(\frac{d\theta}{dt}\right)^2+n^2\frac{\theta^3}{6}-\dots.$$

Since κ and θ are very small, we might at first suppose that it would be sufficient to take as a first approximation

$$\frac{d^2\theta}{dt^2}+n^2\theta=0.$$

This gives $\theta=\alpha\sin nt$, the origin of measurement of t being so chosen that t and θ vanish together. If we substitute this in the small terms we get

$$\frac{d^2\theta}{dt^2}+n^2\theta=-2\kappa n\,.\,\alpha\cos nt+\&c.,$$

which gives $\theta=\alpha\sin nt-\kappa\alpha\,.\,t\sin nt+\&c.$

This additional term contains t as a factor, and shows that our first approximation was not sufficiently near the truth to represent the motion except for a short time.

To obtain a sufficiently near first approximation we must include in it the small term $2\kappa\frac{d\theta}{dt}$, we have therefore

$$\frac{d^2\theta}{dt^2}+2\kappa\frac{d\theta}{dt}+n^2\theta=0.$$

This gives

$$\theta=\alpha e^{-\kappa t}.\sin mt,$$

where for the sake of brevity we have put $n^2-\kappa^2=m^2$.

In our second approximation we shall reject all terms of the order α^3 or $\alpha^2\kappa$ unless they are such that after integration they rise in importance in the manner explained in Art. 201. We thus get

$$\begin{aligned}\frac{d^2\theta}{dt^2}+2\kappa\frac{d\theta}{dt}+n^2\theta=&-\frac{\mu\alpha^2m^2}{2}e^{-2\kappa t}(1+\cos 2mt)\\&+\frac{n^2}{6}\alpha^3\frac{e^{-3\kappa t}}{4}(3\sin mt-\sin 3mt)\\&-\mu\alpha^2\kappa e^{-2\kappa t}\left(-\frac{\kappa}{2}+\frac{\kappa}{2}\cos 2mt+m\sin 2mt\right),\end{aligned}$$

where all the terms after those in the first line are of the third order, and are to be rejected unless they rise in importance.

To solve this, let us first consider the general case

$$\frac{d^2\theta}{dt^2}+2\kappa\frac{d\theta}{dt}+n^2\theta=e^{-p\kappa t}.(A\sin rmt+B\cos rmt).$$

Put $\theta=e^{-p\kappa t}(L\sin rmt+M\cos rmt)$.

Substituting we get

$$\left.\begin{aligned}L\{(p-1)^2\kappa^2+m^2(1-r^2)\}+2(p-1)\kappa rmM&=A\\M\{(p-1)^2\kappa^2+m^2(1-r^2)\}-2(p-1)\kappa rmL&=B\end{aligned}\right\}.$$

Now κ is very small. If then r be not equal to unity, we have nearly

$$L=\frac{A}{m^2(1-r^2)},\quad M=\frac{B}{m^2(1-r^2)};$$

but if $r=1$, we have nearly

$$L=\frac{-B}{2(p-1)\kappa m},\quad M=\frac{A}{2(p-1)\kappa m}.$$

The case of $p=1$ does not occur in our problem.

It appears that those terms only in the differential equation which have $r=1$ give rise to terms in the value of x which have the small quantity κ in the denominator. Hence in the differential equation the only term of the third order which should be retained is the first. We thus find, putting successively $r=0$, $r=2$, $r=1$,

$$\theta=ae^{-\kappa t}\sin mt-\frac{\mu a^2}{2}e^{-2\kappa t}+\frac{\mu a^2}{6}e^{-2\kappa t}\cos 2mt+\frac{n^2a^3}{32\kappa m}e^{-3\kappa t}\cos mt.$$

This equation determines the motion only during any one swing of the pendulum; when the pendulum turns to go back μ changes sign. Let us suppose the pendulum to be moving from left to right, and let us find the lengths of the arcs of descent and ascent. To do this, we must put $\frac{d\theta}{dt}=0$. Let the equation be written in the form $\theta=f(t)$, then if we neglect all the small terms, $\frac{d\theta}{dt}$ vanishes when $mt=\pm\frac{\pi}{2}$. Put then $mt=-\frac{\pi}{2}+x$ where x is a small quantity, we have

$$f'(t)=f'\left(-\frac{\pi}{2m}\right)+f''\left(-\frac{\pi}{2m}\right)\frac{x}{m}=0.$$

Now $f'(t)=ae^{-\kappa t}(m\cos mt-\kappa\sin mt)$

$$-\frac{\mu a^2}{2}e^{-2\kappa t}\left(-2\kappa+\frac{2\kappa}{3}\cos 2mt+\frac{2m}{3}\sin 2mt\right)$$

$$+\frac{n^2a^3}{32\kappa m}e^{-3\kappa t}(-m\sin mt-3\kappa\cos mt).$$

A sufficiently near approximation to the value of $f''(t)$ may be found by differentiating the first line of the value of $f'(t)$. We thus find

$$x=-\frac{\kappa}{m}-\frac{4}{3}\frac{\mu a\kappa}{m}-\frac{n^2a^2}{32\kappa m};$$

the second of these terms being smaller than the other two might be neglected. We also find as the arc of descent

$$\theta = f\left(-\frac{\pi}{2m}\right) + f'\left(-\frac{\pi}{2m}\right)\frac{x}{m}$$

$$= -\left[\alpha e^{\frac{\kappa\pi}{2m}} + \frac{2}{3}\mu\alpha^2 e^{\frac{\kappa\pi}{m}} - x\left\{\kappa\alpha e^{\frac{\kappa\pi}{m}} + \frac{n^2\alpha^3}{32\kappa m}e^{\frac{3\kappa\pi}{2m}}\right\}\right].$$

Similarly to find the arc of ascent we put $mt = \frac{\pi}{2} + y$. This gives $y = -\frac{\kappa}{m} - \frac{n^2\alpha^2}{32\kappa m}$, and the arc of ascent is

$$\theta = \alpha e^{-\frac{\kappa\pi}{2m}} - \frac{2}{3}\mu\alpha^2 e^{-\frac{\kappa\pi}{m}} - y\left\{\kappa\alpha e^{-\frac{\kappa\pi}{2m}} + \frac{n^2\alpha^3}{32\kappa m}e^{-\frac{3\kappa\pi}{2m}}\right\}.$$

In these expressions for the arcs of descent and ascent the terms containing x and y are very small, and assuming κ not to be extremely small, these terms will be neglected*.

Now α is different for every swing of the pendulum, we must therefore eliminate α. Let u_n and u_{n+1} be two successive arcs of descent and ascent, and let $\lambda = e^{-\frac{\kappa\pi}{2m}}$, so that λ is a little less than unity. Then we have

$$\left.\begin{aligned} u_n &= \alpha\frac{1}{\lambda} + \frac{2}{3}\mu\alpha^2\frac{1}{\lambda^2} \\ u_{n+1} &= \alpha\lambda - \frac{2}{3}\mu\alpha^2\lambda^2 \end{aligned}\right\};$$

$$\left.\begin{aligned} \therefore\ u_{n+1} - \lambda^2 u_n &= -\frac{2}{3}\mu\alpha^2(1+\lambda^2) \\ u_n \,.\, u_{n+1} &= \alpha^2 + \frac{2}{3}\mu\alpha^3\frac{1-\lambda^2}{\lambda} + \ldots \end{aligned}\right\}.$$

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* If these terms are not neglected the equation connecting the successive arcs of descent and ascent becomes

$$\frac{1}{u_n} - \frac{\lambda^2}{u_{n+1}} = -\frac{2}{3}\mu(1+\lambda^2) + \frac{n^2x}{32\kappa m}\,\frac{1-\lambda^4}{\lambda},$$

Now $1-\lambda^4 = \frac{2\kappa\pi}{m}$ nearly, so that this additional term is very small compared with that retained.

Hence
$$\frac{1}{u_n} - \frac{\lambda^2}{u_{n+1}} = -\frac{2}{3}\mu(1+\lambda^2).$$

This equation connects the successive arcs of descent and ascent. It may be written in the form
$$\frac{1}{u_{n+1}} + \frac{1}{c} = \frac{1}{\lambda^2}\left(\frac{1}{u_n} + \frac{1}{c}\right),$$
where
$$c = \frac{3}{2\mu}\frac{1-\lambda^2}{1+\lambda^2} = \frac{3\kappa\pi}{4\mu m} \text{ nearly.}$$

The successive arcs are, therefore, such that $\frac{1}{u_n} + \frac{1}{c}$ is the general term of a geometrical series whose ratio is $e^{\frac{\kappa\pi}{m}}$. The ratio of any arc u_n to the following arc u_{n+1} is
$$\frac{u_n}{u_{n+1}} = e^{\frac{\kappa\pi}{m}} + \frac{u_n}{c}\left(e^{\frac{\kappa\pi}{m}} - 1\right),$$
which continually decreases with the arc. In any series of oscillations the ratio is at first greater and afterwards less than its mean value. This result seems to agree with experiment.

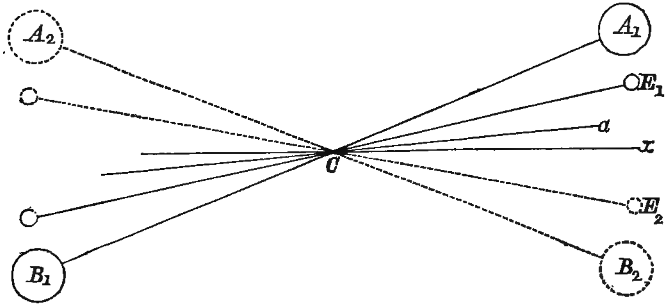
To find the time of oscillation. Let t_1, t_2 be the times at which the pendulum is at the extreme left and right of its arc of oscillation. Then
$$\left.\begin{aligned} mt_1 &= -\frac{\pi}{2} - \frac{\kappa}{m} - \frac{n^2\alpha^2}{32m\kappa} \\ mt_2 &= \frac{\pi}{2} - \frac{\kappa}{m} - \frac{n^2\alpha^2}{32m\kappa} \end{aligned}\right\};$$
$$\therefore \text{ time} = t_2 - t_1 = \frac{\pi}{m}.$$

This result is independent of the arc, so that the time of oscillation remains constant throughout the motion. The time is however not exactly the same as in vacuo, but is a little longer; the difference depending on the *square* of the small quantity κ.

The Cavendish Experiment.

203. As an example of the mode in which the theory of small oscillations may be used as a means of discovery we have selected the Cavendish Experiment. The object of this experiment is to compare the mass of the earth with that of some given body. The plan of effecting this by means of a torsion-rod was first suggested by the Rev. John Michell. As he died before he had time to enter on the experiments, his plan was taken up by Mr Cavendish, who published the result of his labours in the *Phil. Trans.* for 1798. His experiments being few in number, it was thought proper to have a new determination. Accordingly in 1837, Mr. Baily was requested by the Astronomical Society to undertake the duty. Mr Baily made upwards of two thousand experiments with balls of different weights and sizes, and suspended in a variety of ways, a full account of which is given in the *Memoirs of the Astronomical Society*, Vol. XIV. The experiments were, in general, conducted in the following manner.

Two small equal balls were attached to the extremities of a fine rod called the torsion-rod, and the rod itself was suspended by a string fixed to its middle point C. Two large spherical masses A, B were fastened on the ends of a plank which could turn freely about its middle point O. The point O was vertically under C and so placed that the four centres of gravity of the four balls were in one horizontal plane.



First, suppose the plank to be placed at right angles to the torsion-rod, then the rod will take up some position of equilibrium called the neutral position, in which the string has no torsion. Let this be represented in the figure by $C\alpha$. Now let the masses A and B be moved round O into some position B_1A_1, making a not very large angle with the neutral position of the torsion-rod. The attractions of the masses A and B on the balls will draw the torsion-rod out of its neutral position into a new position of equilibrium, in which the attraction is balanced by the torsion of the string. Let this be represented in the figure by CE_1. The angle of deviation $E_1C\alpha$ and the time of oscillation of the rod about this position of equilibrium might be observed.

Secondly, replace the plank AB at right angles to the neutral position of the rod, and move it in the opposite direction until the masses A and B come into some position A_2B_2 near the rod but on the side opposite to B_1A_1. Then the torsion-rod will perform oscillations about another position of equilibrium CE_2 under the influence of the attraction of the masses and the torsion of the string. As before, the time of oscillation and the deviation $E_2C\alpha$ might be observed.

In order to eliminate the errors of observation, this process was repeated over and over again, and the mean results taken. The positions B_1A_1 and A_2B_2, into which the masses were alternately put, were as nearly as possible the same throughout all the experiments. The neutral position $C\alpha$ of the rod very nearly bisected the angle between B_1A_1 and A_2B_2, but as this neutral position, possibly owing to changes in the torsion of the string, was found to undergo slight changes of position, it is not to be considered in any one experiment coincident with the bisector of the angle A_1CB_2.

Let Cx be any line fixed in space from which the angles may be measured. Let b be the angle $xC\alpha$, which the neutral position of the rod makes with Cx; A and B the angles which the alternate positions, B_1A_1 and A_2B_2, of the straight line joining the centres of the masses, make with Cx; and let $a = \dfrac{A+B}{2}$. Also let x be the angle which the torsion-rod makes with Cx at the time t.

Supposing the masses to be in the position A_1B_1, the moment about CO of their attraction on the two balls and on the rod will be a function only of the angle between the rod and the line A_1B_1; let this moment be represented by $\phi(A-x)$. The whole apparatus was enclosed in a wooden casing to protect it from any currents of air. The attraction of this casing cannot be neglected. As it may be different in different positions of the rod, let the moment of its attraction about CO be $\psi(x)$. Also the torsion of the string will very nearly be proportional to the angle through which it has been twisted. Let its moment about CO be $E(x-b)$.

If then I be the moment of inertia of the balls and rod about the axis CO, the equation of motion will be

$$I\frac{d^2x}{dt^2} = \phi(A-x) + \psi(x) - E(x-b).$$

Now $a-x$ is a small quantity, let it be represented by ξ. Substituting for x and expanding by Taylor's theorem in powers of ξ, we get

$$-I\frac{d^2\xi}{dt^2} = \phi(A-a) + \psi(a) - E(a-b)$$

$$+\{\phi'(A-a) - \psi'(a) + E\}\,\xi.$$

Let
$$\left.\begin{aligned} n^2 &= \frac{\phi'(A-a) - \psi'(a) + E}{I} \\ e &= a + \frac{\phi(A-a) + \psi(a) - E(a-b)}{In^2} \end{aligned}\right\}.$$

Then
$$x = e + L\sin(nt + L'),$$

where L and L' are two arbitrary constants. We see therefore that in the position of equilibrium the angle the torsion-rod makes with the axis of x is e, and the time of oscillation about the position of equilibrium is $\frac{2\pi}{n}$.

Let us now suppose the masses to be moved into their alternate position A_2B_2; the moment of their attraction on the balls and rod will now be $-\phi(x-B)$. The equation of motion is therefore

$$I\frac{d^2x}{dt^2}=-\phi(x-B)+\psi(x)-E(x-b).$$

Let $a=x-\xi$, then substituting for B its value $2a-A$, we find by the same reasoning as before

$$x=e'+N\sin(nt+N'),$$

where n has the same value as before and

$$e'=a+\frac{-\phi(A-a)+\psi(a)-E(a-b)}{In^2}.$$

In these expressions, the attraction $\psi(a)$ of the casing, the coefficient of torsion E and the angle b are all unknown. But they all disappear together, if we take the difference between e and e'. We then find

$$\frac{\phi(A-a)}{I}=\frac{e-e'}{2}\cdot\left(\frac{2\pi}{T}\right)^2 \text{ .... (A),}$$

where T is the time of a complete oscillation of the torsion-rod about either of the *disturbed* positions of equilibrium. Thus the attraction $\phi(A-a)$ can be found if the angle $e-e'$ between the two positions of equilibrium and also the time of oscillation about either can be observed.

The function $\phi(A-a)$ is the moment of the attraction of the masses and the plank on the balls and rod, when the rod has been placed in a position Cf, bisecting the angle A_1CB_2 between the alternate positions of the masses. Let M be the mass of either of the masses A and B, m that of one of the small balls, m' that of the rod. Let the attraction of M on m be represented by $\mu\frac{Mm}{D^2}$, where D is the distance between their centres. If (p, q) be the co-ordinates of the centre of A_1 referred to Cf as the axis of x, the moment about C of the attraction of both the masses on both the balls is

$$=2\mu Mm\left\{\frac{cq}{\{(p-c)^2+q^2\}^{\frac{3}{2}}}-\frac{cq}{\{(p+c)^2+q^2\}^{\frac{3}{2}}}\right\},$$

where c is the distance of the centre of either ball a, b from the centre C of motion. Let this be represented by μMmP. The moments of the attraction of the masses on the rod may by integration be found $= \mu Mm'Q$, where Q is a known function of the linear dimensions of the apparatus. The attraction of the plank might also be taken account of. Thus we find

$$\phi\,(A - a) = \mu M\,(mP + m'Q).$$

If r be the radius of either ball, we have

$$I = 2m\left\{c^2 + \frac{2}{5}r^2\right\} + m'\frac{(c-r)^2}{3},$$

which may be represented by

$$I = mP' + m'Q',$$

where P' and Q' are known functions of the linear dimensions of the rod and balls. Hence we find by substituting in equation (A)

$$\mu M \,.\, \frac{mP + m'Q}{mP' + m'Q'} = \frac{e - e'}{2} \,.\left(\frac{2\pi}{T}\right)^2.$$

Let E be the mass of the earth, R its radius and g the force of gravity, then

$$g = \mu \frac{E}{R^2}.\text{*}$$

Substituting for μ, we find

$$\frac{M}{E} = \frac{e - e'}{2}\,.\left(\frac{2\pi}{T}\right)^2.\,\frac{1}{gR^2}\,.\,\frac{\frac{m}{m'}P' + Q'}{\frac{m}{m'}\,P + Q}.$$

The ratio $\frac{m}{m'}$ was taken equal to the ratio of the weights of the ball and rod *weighed in vacuo*, but it would clearly

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[*] In Baily's experiment, a more accurate value of g was used. If ϵ be the ellipticity of the earth, m the ratio of centrifugal force at the equator to equatoreal gravity, and λ the latitude of the place, we have

$$g = \mu \frac{E}{R^2}\{1 - 2\epsilon + \left(\frac{5}{2}m - \epsilon\right)\cos^2\lambda\}.$$

have been more accurate to have taken it equal to the ratio weighed in air. For since the masses attract the air as well as the balls, the pressure of the air on the side of a ball nearest the attracting mass is greater than that on the furthest side. The difference of these pressures is equal to the attraction of the mass on the air displaced by the ball.

By this theory the discovery of the mass of the earth has been reduced to the determination of two elements, (1) the time of oscillation of the torsion-rod, and (2) the angle $e - e'$ between its two positions of equilibrium when under the influence of the masses in their alternate positions. To observe these, a small mirror was attached to the rod at C with its plane nearly perpendicular to the rod. A scale was engraved on a vertical plate at a distance of 108 inches from the mirror, and the image of the scale formed by reflection on the mirror was viewed in a telescope placed just over the scale. The telescope was furnished with three vertical wires in its focus. As the torsion-rod turned on its axis, the image of the scale was seen in the telescope to move horizontally across the wires, and at any instant the number of the scale coincident with the middle wire constituted the reading. The scale was divided by vertical lines one-thirteenth of an inch apart and numbered from 20 to 180 to avoid negative readings. The angle turned through by the rod when the image of the scale moved through a space corresponding to the interval of two divisions was therefore $\frac{1}{13} \cdot \frac{1}{108} \cdot \frac{1}{2} = 73'' \cdot 46$. But the division lines were cut diagonally and subdivided decimally by horizontal lines; so that not only could the tenth of a division be clearly distinguished, but after some little practice, the fractional parts of these tenths. The arc of oscillation of the torsion-rod was so small that the square of its circular measure could be neglected; but as it extended over several divisions it is clear that it could be observed with accuracy. A minute description of the mode in which the observations were made would not find a fit place in a treatise on Dynamics, we must therefore refer the reader to Baily's Memoir.

In this investigation no notice has been taken of the effect of the resistance of the air on the arc of vibration. This was, to some extent at least, eliminated by a peculiar mode of taking the means of the observations. In this way also some allowance was made for the motion of the neutral position of the torsion-rod.

The final result was that the mean density of the earth is 5·6747.

EXAMPLES.

1. A uniform rod of length $2c$ rests in stable equilibrium with its lower end at the vertex of a cycloid whose plane is vertical and vertex downwards, and passes through a small smooth fixed ring situated in the axis at a distance b from the vertex. Show that if the equilibrium be slightly disturbed, the rod will perform small oscillations with its lower end on the arc of the cycloid in the time $4\pi\sqrt{\dfrac{a\{c^2+3(b-c)^2\}}{3g(b^2-4ac)}}$, where $2a$ is the length of the axis of the cycloid.

2. A small smooth ring slides on a circular wire of radius a which is constrained to revolve about a vertical axis in its own plane, at a distance c from the centre of the ring, with a uniform angular velocity $\sqrt{\dfrac{g\sqrt{2}}{c\sqrt{2}+a}}$; show that the ring will be in a position of stable relative equilibrium when the radius of the circular wire passing through it is inclined at an angle 45° to the horizon; and that if the ring be slightly displaced, it will perform a small oscillation in the time

$$2\pi\left\{\frac{a\sqrt{2}}{g}\cdot\frac{c\sqrt{2}+a}{c\sqrt{8}+a}\right\}^{\frac{1}{2}}.$$

3. A uniform bar of length $2a$, suspended by two equal parallel strings each of length $2b$ from two points in the same horizontal line is turned through a small angle about the vertical line through the middle point, show that the time of a small oscillation is $2\pi\sqrt{\dfrac{bk^2}{ga^2}}$.

4. The upper extremity of a uniform beam of length $2a$ is constrained to slide on a smooth horizontal rod without

inertia, and the lower along a smooth vertical rod through the upper extremity of which the horizontal rod passes: the system rotates freely about the vertical rod, prove that if α be the inclination of the beam to the vertical when in a position of relative equilibrium, the angular velocity of the system will be $\left(\frac{3g}{4a\cos\alpha}\right)^{\frac{1}{2}}$, and if the beam be slightly displaced from this position show that it will make a small oscillation in the time

$$\frac{4\pi}{\left\{\frac{3g}{a}(\sec\alpha + 3\cos\alpha)\right\}^{\frac{1}{2}}}.$$

5. Two equal heavy rods connected by a hinge which allows them to move in a vertical plane rotate uniformly about a vertical axis through the hinge, and a string whose length is twice that of either rod is fastened to their extremities and bears a weight at its middle point. If M, M' be the masses of a rod and the particle, and $2a$ the length of a rod, prove that the angular velocity about the vertical axis when the rods and string form a square is $\sqrt{\frac{3g}{2a\sqrt{2}} \cdot \frac{M+2M'}{M}}$, and if the weight be slightly depressed in a vertical direction the time of a small oscillation is $2\pi\sqrt{\frac{4a(3+\sqrt{2})}{15g} \cdot \frac{M}{M+2M'}}$.

6. A ring of weight W which slides on a rod inclined to the vertical at an angle α is attached by means of an elastic string to a point in the plane of the rod so situated that its least distance from the rod is equal to the natural length of the string. Prove that if θ be the inclination of the string to the rod when in equilibrium, $\cot\theta - \cos\theta = \frac{W}{w}\cos\alpha$, where w is the modulus of elasticity of the string. And if the ring be slightly displaced the time of a small oscillation will be $2\pi\sqrt{\frac{Wl}{wg}\,\frac{1}{1-\sin^3\theta}}$, where l is the natural length of the string.

7. A circular tube of radius a contains an elastic string fastened at its highest point equal in length to $\frac{1}{8}$ of its circum-

ference, and having attached to its other extremity a heavy particle which hanging vertically would double its length. The system revolves about the vertical diameter with an angular velocity $\sqrt{\frac{g}{a}}$. Find the position of relative equilibrium and prove that if the particle be slightly disturbed the time of a small oscillation is $\frac{2\pi\sqrt{\pi}}{\sqrt{\pi+4}}\sqrt{\frac{a}{g}}$.

8. A heavy uniform rod AB has its lower extremity A fixed to a vertical axis and an elastic string connects B to another point C in the axis such that $AC = \frac{AB}{\sqrt{2}} = a$; the whole is made to revolve round AC with such angular velocity that the string is double its natural length, and horizontal when the system is in relative equilibrium and then left to itself. If the rod be slightly disturbed in a vertical plane, prove that the time of a small oscillation is $2\pi\sqrt{\frac{4a}{21g}}$, the weight of the rod being sufficient to stretch the string to twice its length.

9. Three equal elastic strings AB, BC, CA surround a circular arc, the end A being fixed. At B and C two equal particles of mass m are fastened. If l be the natural length of each string supposed always stretched and λ the modulus of elasticity, show that if the equilibrium be disturbed the particles will be at their original distance after intervals $\pi\sqrt{\frac{ml}{\lambda}}$.

10. A particle of mass M is placed near the centre of a smooth circular horizontal table of radius a, strings are attached to the particle and pass over n smooth pullies which are placed at equal intervals round the circumference of the circle; to the other end of each of these strings a particle of mass M is attached; show that the time of a small oscillation of the system is $2\pi\left(\frac{2+n}{n}\frac{a}{g}\right)^{\frac{1}{2}}$.

11. In a circular tube of uniform bore containing air, slide two circular discs exactly fitting the tube. The two

discs are placed initially so that the line joining their centres passes through the centre of the tube, and the air in the tube is initially of its natural density. One disc is projected so that the initial velocity of its centre is a small quantity w. If the inertia of the air be neglected, prove that the point on the axis of the tube equidistant from the centre of the discs moves uniformly and that the time of an oscillation of each disc is $2\pi\sqrt{\frac{Ma}{4P}}$, where M is the mass of each disc, a the radius of the axis of tube, P the pressure of air on the disc in its natural state.

12. A uniform beam of mass M and length $2a$ can turn round a fixed horizontal axis at one end; to the other end of the beam a string of length l is attached and at the other end of the string a particle of mass m. If, during a small oscillation of the system, the inclination of the string to the vertical is always twice that of the beam, then

$$Mk^2 = (M - 2m)\,a\,(a + l).$$

13. A conical surface of semivertical angle α is fixed with its axis inclined at an angle θ to the vertical, and a smooth cone of semivertical angle β is placed within it so that the vertices coincide. Show that the time of a small oscillation $= 2\pi\sqrt{\frac{a \sin(\alpha - \beta)}{g \sin\theta}}$, where a is the distance of the centre of gravity of the cone from the vertex.

14. A number of bodies, the particles of which attract each other with forces varying as the distance, are capable of motion on certain curves and surfaces. Prove that if A, B, C be the moments of inertia of the system about three axes mutually at right angles through its centre of gravity, the positions of stable equilibrium will be found by making $A + B + C$ a minimum.

15. A rigid body is suspended by two equal and parallel threads attached to it at two points symmetrically situated with respect to a principal axis through the centre of gravity which is vertical, and being turned round that axis through a small angle is left to perform small *finite* oscillations. Investigate the reduction to infinitely small oscillations.

CHAPTER IX.

MOTION OF A BODY UNDER THE ACTION OF NO FORCES.

204. *To determine the motion of a body about a fixed point, in the case in which there are no impressed forces.*

The equations of motion are by Art. 107,

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 = 0 \\ B\frac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1 = 0 \\ C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 = 0 \end{aligned}\right\};$$

multiplying these respectively by ω_1, ω_2, ω_3; adding and integrating, we get

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = T \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where T is an arbitrary constant.

Again, multiplying the equations respectively by $A\omega_1$, $B\omega_2$, $C\omega_3$ we get, similarly,

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = G^2 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where G is an arbitrary constant.

To find a third integral, let

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2 \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

$$\therefore\ \omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt} + \omega_3\frac{d\omega_3}{dt} = \omega\frac{d\omega}{dt};$$

then multiplying the original equations respectively by $\frac{\omega_1}{A}$, $\frac{\omega_2}{B}$, $\frac{\omega_3}{C}$, and adding, we get

$$\omega\frac{d\omega}{dt}=\left(\frac{B-C}{A}+\frac{C-A}{B}+\frac{A-B}{C}\right)\omega_1\omega_2\omega_3 \dots\dots\dots\dots (4).$$

$$=-\frac{(B-C)(C-A)(A-B)}{ABC}\omega_1\omega_2\omega_3.$$

But solving the equations (1), (2), (3), we get

$$\left.\begin{aligned}\omega_1^2&=\frac{BC}{(A-C)(A-B)}\cdot(-\lambda_1+\omega^2)\\ \omega_2^2&=\frac{CA}{(B-A)(B-C)}\cdot(-\lambda_2+\omega^2)\\ \omega_3^2&=\frac{AB}{(C-B)(C-A)}\cdot(-\lambda_3+\omega^2)\end{aligned}\right\}\dots\dots\dots\dots (5),$$

where $\lambda_1=\frac{T(B+C)-G^2}{BC}$, with similar expressions for λ_2 and λ_3. Substituting in equation (4), we have

$$\omega\frac{d\omega}{dt}=\sqrt{(\lambda_1-\omega^2)(\lambda_2-\omega^2)(\lambda_3-\omega^2)}\dots\dots\dots\dots (6),$$

The integration of equation (6) can be reduced without difficulty to depend on an elliptical integral. The integration can be effected in finite terms in two cases; when $A=B$, and when $G^2=TB$, where B is neither the greatest nor the least of the three quantities A, B, C. Both these cases will be discussed further on.

205. The integrals (1) and (2) of the last article are the fundamental equations from which much of what follows may be deduced. It is therefore worth while to show that they may also be deduced from the principles of Conservation of Areas and Vis Viva. Suppose the body to be set in motion by an impulsive couple whose magnitude is G. Then we know by Art. 40 that throughout the whole of the subse-

quent motion, the moment of the momentum about every straight line is constant and equal to the moment of the couple G about that line. Now by Art. 118 the moments of the momentum about the principal axes at any instant are respectively $A\omega_1$, $B\omega_2$, $C\omega_3$. Let (α, β, γ) be the direction angles of the normal to the plane of the couple G referred to the principal axes as co-ordinate axes. Then we have

$$\left.\begin{aligned} A\omega_1 &= G\cos\alpha \\ B\omega_2 &= G\cos\beta \\ C\omega_3 &= G\cos\gamma \end{aligned}\right\}\dots\dots\dots\dots\dots\dots\dots (1),$$

and by adding the squares of these three equations,

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = G^2 \dots\dots\dots\dots\dots\dots (2).$$

Throughout the whole of the subsequent motion, the whole momentum of the body is equivalent to the couple G. It is therefore clear that if at any instant the body were acted on by an impulsive couple equal and opposite to the couple G, it would be reduced to rest.

206. It follows also from Art. 138 that the plane of this couple is the Invariable plane, and the normal to it is the Invariable line. This line is absolutely fixed in space*, and the equations (1) give the direction cosines of a line fixed in space with reference to axes moving with the body.

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* That the line whose direction cosines, referred to the moving principal axes, are $\frac{A\omega_1}{G}$, $\frac{B\omega_2}{G}$, $\frac{C\omega_3}{G}$, is absolutely fixed in space may be also proved thus. Let (x, y, z) be the co-ordinates of any point in it distant r from the origin, then its actual velocity in space resolved parallel to the instantaneous position of the axis of x

$$= \frac{dx}{dt} - y\omega_3 + z\omega_2,$$

$$= \frac{d}{dt}\left(r\frac{A\omega_1}{G}\right) - r\frac{B\omega_2}{G}\omega_3 + r\frac{C\omega_3}{G}\omega_2,$$

$$= \frac{r}{G}\left\{A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3\right\}$$

$$= 0.$$

Similarly the velocities parallel to the other axes are zero.

It appears from these equations that if the direction cosines of the initial axis of rotation be (l, m, n) when referred to the principal axes of the body, then the direction cosines of the invariable plane are proportional to Al, Bm, Cn. If the axes of reference be not the principal axes of the body at the fixed point, the equations corresponding to (1) and (2) may be deduced from Arts. 138 and 118.

207. Since the body moves under the action of no impressed forces, we know that the Vis Viva will be constant throughout the motion. Hence by Art. 158, we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = T \dots\dots\dots\dots\dots\dots (3),$$

where T* is a constant to be determined from the initial values of ω_1, ω_2, ω_3.

The equations (1), (2), (3) will suffice to determine the path in space described by every particle of the body, but not the position at any given time.

208. It will generally be supposed that A, B, C are in order of magnitude, so that A is greater than B, and B than C. The axis of B will be called the axis of mean moment. If we eliminate ω_1 from the equations (2) and (3), we have

$$AT - G^2 = B(A - B)\,\omega_2^2 + C(A - C)\,\omega_3^2,$$

which is essentially positive. In the same way we can show that $CT - G^2$ is negative. Thus the quantity $\dfrac{G^2}{T}$ may have any value lying between the greatest and least moments of inertia.

The three quantities λ_1, λ_2, λ_3 in Art. 204 are all positive quantities; for since $B + C - A$ is positive, and $\dfrac{G^2}{T} < A$, it follows that λ_1 is positive. The others may be proved to be positive in the same way. Also $\lambda_1 - \lambda_2 = \dfrac{TC - G^2}{ABC}(A - B)$ with similar expressions for $\lambda_2 - \lambda_3$ and $\lambda_3 - \lambda_1$. It easily follows that λ_2 is the greatest of the three, and λ_1 or λ_3 is the least according as $\frac{G^2}{T}$ is > or < B.

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* It should be observed that in this Chapter T represents the whole vis viva of the body. In treating of Lagrange's equations in Chapter VII. it was convenient to let T represent *half* the vis viva of the system.

209. *To explain Poinsot's representation of the motion by means of the momental ellipsoid.*

Let the momental ellipsoid at the fixed point be constructed, and let its equation be

$$Ax^2 + By^2 + Cz^2 = \epsilon^4.$$

Let r be the radius vector of this ellipsoid coinciding with the instantaneous axis, and p the perpendicular from the centre on the tangent plane at the extremity of r. Also let ω be the angular velocity about the instantaneous axis.

The equations to the instantaneous axis are

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3},$$

and if (x, y, z) be the co-ordinates of the extremity of the length r, each of these fractions is equal to $\frac{r}{\omega}$.

Substituting in the equation to the ellipsoid, we have

$$(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)\frac{r^2}{\omega^2} = \epsilon^4,$$

$$\therefore \omega = \frac{\sqrt{T}}{\epsilon^2} . r.$$

Again the expression for the perpendicular on the tangent plane at (x, y, z) is known to be

$$\frac{1}{p^2} = \frac{A^2x^2 + B^2y^2 + C^2z^2}{\epsilon^8},$$

hence substituting as before we get

$$\frac{1}{p^2} = \frac{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}{\epsilon^8} \cdot \frac{r^2}{\omega^2}$$

$$= \frac{G^2}{\epsilon^8} \cdot \frac{\epsilon^4}{T},$$

$$\therefore p = \frac{\sqrt{T}}{G} \cdot \epsilon^2.$$

The equation to the tangent plane at the point (x, y, z) is

$$Ax\xi + By\eta + Cz\zeta = \epsilon^4,$$

substituting for (x, y, z) we see that the equations to the perpendicular from the origin are

$$\frac{\xi}{A\omega_1} = \frac{\eta}{B\omega_2} = \frac{\zeta}{C\omega_3};$$

but these are the equations to the invariable line. Hence this perpendicular is fixed in space.

From these equations we infer

(1) *The angular velocity about the radius vector round which the body is turning varies as that radius vector.*

(2) *The resolved part of the angular velocity about the perpendicular on the tangent plane at the extremity of the instantaneous axis is constant.*

For the cosine of the angle between the perpendicular and the radius vector is $= \frac{p}{r}$. Hence the resolved angular velocity is $= \omega \frac{p}{r} = \frac{T}{G}$, which is constant.

(3) *The perpendicular on the tangent plane at the extremity of the instantaneous axis is fixed in direction,* viz. *normal to the invariable plane and constant in length.*

The motion of the momental ellipsoid is therefore such that its centre being fixed, it always touches a fixed plane, and the point of contact being in the instantaneous axis has no velocity. *Hence the motion may be represented by supposing the momental ellipsoid to roll on the fixed plane with its centre fixed.*

210. To assist our conception of the motion of the body, let us suppose it so placed, that the plane of the couple G, which would set it in motion, is horizontal. Let a tangent

plane to the momental ellipsoid be drawn parallel to the plane of the couple G, and let this plane be fixed in space. Let the ellipsoid roll on this fixed plane, its centre remaining fixed, with an angular velocity which varies as the radius vector, to the point of contact, and let it carry the given body with it. We shall then have constructed the motion which the body would have assumed if it had been left to itself after the initial action of the impulsive couple G.*

The point of contact of the ellipsoid with the plane on which it rolls traces out two curves, one on the surface of the ellipsoid, and one on the plane. The first of these is fixed in the body and is called the *polhode*, the second is fixed in space and is called the *herpolhode*. The equations to any polhode referred to the principal axes of the body may be found from the consideration that the length of the perpendicular on the tangent plane to the ellipsoid at any point of the polhode is constant. Hence its equations are

$$\left.\begin{aligned} A^2x^2 + B^2y^2 + C^2z^2 &= \frac{G^2\epsilon^4}{T} \\ Ax^2 + By^2 + Cz^2 &= \epsilon^4 \end{aligned}\right\}.$$

Eliminating y, we have

$$A(A-B)x^2 + C(C-B)z^2 = \left(\frac{G^2}{T} - B\right)\epsilon^4.$$

Hence if B be the axis of greatest or least moment of inertia, the signs of the coefficients of x^2 and z^2 will be the same, and the projection of the polhode will be an ellipse. But if B be the axis of mean moment of inertia, the projection is an hyperbola.

A polhode is therefore a closed curve drawn round the axis of greatest or least moment, and the concavity is turned towards the axis of greatest or least moment according as $\frac{G^2}{T}$ is greater or less than the mean moment of inertia. The boundary line which separates the two sets of polhodes is that polhode whose projection on the plane perpendicular to the axis of mean moment is an hyperbola whose concavity is turned neither to the axis of greatest, nor to the axis of least moment. In this case $\frac{G^2}{T} = B$, and the projection consists of two straight lines whose equations are

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* Prof. Sylvester has pointed out a *dynamical* relation between the free rotating body and the ellipsoidal top, as he calls Poinsot's central ellipsoid. If a *material* ellipsoidal top be constructed of uniform density, similar to Poinsot's central ellipsoid, and if with its centre fixed it be set rolling on a perfectly rough horizontal plane, it will represent the motion of the free rotating body not in space only, but also in time: the body and the top may be conceived as continually moving round the same axis and at the same rate at each moment of time. The reader is referred to the memoir in the *Philosophical Transactions* for 1866.

$$A\,(A - B)\,x^2 - C\,(B - C)\,z^2 = 0.$$

This polhode consists of two ellipses passing through the axis of mean moment, and corresponds to the case in which the perpendicular on the tangent plane is equal to the mean axis of the ellipsoid. This polhode is called the *separating polhode.*

Since the projection of the polhode on one of the principal planes is always an ellipse, the polhode must be a re-entering curve.

By considering the herpolhode to be traced out by the rolling of an ellipsoid on the plane of the paper, it is clear that the herpolhode always lies between two circles which it alternately touches. The herpolhode is therefore not in general re-entering; but if the angular distance of the two points in which it successively touches the same circle be commensurable with 2π, it will be a re-entering curve.

211. The motion of the body may also be represented geometrically by means of the ellipsoid of gyration. This ellipsoid is the reciprocal of the momental ellipsoid, and the motion of the one ellipsoid may be deduced from that of the other by reciprocating the properties proved in the preceding Articles. We find,

(1) The equation to the ellipsoid referred to its principal axes is

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1.$$

(2) This ellipsoid moves so that its superficies always passes through a point fixed in space. This point lies in the invariable line at a distance $\frac{G}{\sqrt{T}}$ from the fixed point. By Art. 208 we know that this distance is less than the greatest, and greater than the least semi-diameter of the ellipsoid.

(3) The perpendicular on the tangent plane at the fixed point is the instantaneous axis of rotation, and the angular velocity of the body varies inversely as the length of this perpendicular. If p be the length of this perpendicular, then $\omega = \frac{\sqrt{T}}{p}$.

(4) The angular velocity about the invariable line is constant and $= \frac{T}{G}$.

The corresponding curve to a polhode is the path described on the moving surface of the ellipsoid by the point fixed in space. This curve is clearly a sphero-conic. The equations to the sphero-conic described under any given initial conditions are easily seen to be

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= \frac{G^2}{T} \\ \frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} &= 1 \end{aligned}\right\}.$$

These sphero-conics may be shown to be closed curves round the axes of greatest and least moment. But in one case, viz. when $\frac{G^2}{T} = B$, where B is neither the greatest nor least moment of inertia, the sphero-conic becomes the two central circular sections of the ellipsoid of gyration.

The motion of the body may thus be constructed by means of either of these ellipsoids. The momental ellipsoid resembles the general shape of the body more nearly than the ellipsoid of gyration. It is protuberant where the body is protuberant, and compressed where the body is compressed. The exact reverse of this is the case in the ellipsoid of gyration.

212. If a body be set in rotation about any principal axis at a fixed point, it will continue to rotate about that axis as a permanent axis. But the three principal axes at the fixed point do not possess equal degrees of stability. If any small disturbing cause act on the body, the axis of rotation will be moved into a neighbouring polhode. If this polhode be a small nearly circular curve enclosing the original axis of rotation, the instantaneous axis will never deviate far in the body from the principal axis which was its original position. The herpolhode also will be a curve of small dimensions, so that the principal axis will never deviate far from a straight line fixed in space. In this case the rotation is said to be *stable.* But if the neighbouring polhode be not nearly circular, the instantaneous axis will deviate far from its original position in the body. In this case a very small disturbance may produce a very great change in the subsequent motion, the rotation is said to be *unstable.*

If the initial axis of rotation be the axis OB of mean moment, the neighbouring polhodes all have their convexities turned towards B. Unless, therefore, the cause of disturbance be such that the axis of rotation is displaced along the separating polhode, the rotation must be unstable. If the displacement be along the separating polhode, the axis may have a tendency to return to its original position. This case will be considered in Art. 228, and for this particular displacement the rotation may be said to be stable.

If the initial axis of rotation be the axis of greatest or least moment, the neighbouring polhodes are ellipses of greater or less eccentricity. If they be nearly circular, the rotation will certainly be stable; if very elliptical, the axis will recede far from its initial position, and the rotation may be called unstable. If OC be the axis of initial rotation, the ratio of the squares of the axes of the neighbouring polhode is ultimately $\frac{A(A-C)}{B(B-C)}$. It is therefore necessary for the stability of the rotation that this ratio should not differ much from unity.

It is well known that the steadiness or stability of a moving body is much increased by a rapid rotation about a principal axis. The reason of this is evident from what precedes. If the body be set rotating about an axis very near the principal axis of greatest or least moment, both the polhode and herpolhode will generally be very small curves; and the direction of that principal axis of the body will be very nearly fixed in space. If now a small impulse f act on the body, the effect will be to alter slightly the position of the instantaneous axis. It will be moved from one polhode to another very near the former, and thus the angular position of the axis in space will not be much affected. Let Ω be the angular velocity of the body, ω that generated by the impulse, then by the parallelogram of angular velocities, the change in the position of the instantaneous axis cannot be greater than $\sin^{-1}\dfrac{\omega}{\Omega}$. If therefore Ω be great, ω must also be great, to produce any considerable change in the axis of rotation. But if the body has no initial rotation Ω, the impulse may generate an angular velocity ω about an axis not nearly coincident with a principal axis. Both the polhode and the herpolhode may then be large curves, and the instantaneous axis of rotation will move about both in the body and in space. The motion will then appear very unsteady. In this manner, for example, we may explain why in the game of cup and ball, spinning the ball about a vertical axis makes it more easy to catch on the spike. Any motion caused by a wrong pull of the string or by gravity, will not produce so great a change of motion as it would have done if the ball had been initially at rest. The fixed direction of the earth's axis in space is also due to its rotation about its axis of figure. In rifles, a rapid rotation is communicated to the bullet about an axis in the direction in which the bullet is moving. It follows, from what precedes, that the axis of rotation will be nearly unchanged throughout the motion. One consequence is that the resistance of the air acts in a known manner on the bullet, the amount of which may therefore be calculated and allowed for.

213. There are four cones which have important relations to the motion of the body.

The First Cone.

As the body moves about the fixed point the invariable line describes a cone in the body. The equations to this line at any moment referred to the principal axes of the body as co-ordinate axes are

$$\frac{x}{A\omega_1} = \frac{y}{B\omega_2} = \frac{z}{C\omega_3},$$

and by Art. 204 we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = (A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2)\frac{T}{G^2};$$

therefore the equation to the cone is

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = (x^2 + y^2 + z^2)\frac{T}{G^2}.$$

This cone is of the second order, its axes coinciding with the principal diameters of the momental ellipsoid. The form of the equation shows that it cuts the ellipsoid of gyration in a sphero-conic, and that its circular sections are parallel to those of the ellipsoid of gyration, and perpendicular to the asymptotes of the focal conic of the momental ellipsoid. It becomes two planes when the initial conditions are such that $\frac{G^2}{T} = B$.

As this cone is generated by the invariable line, it may be called the *Invariable Cone.*

214. *To find the rate at which the invariable line describes its cone in the body.*

Let OI be the radius vector of the momental ellipsoid coinciding with the instantaneous axis, OG the perpendicular on the tangent plane at I. Let a radius vector, which at the time t coincided with the invariable line, make an angle $d\psi$ at the time $t + dt$ with the invariable line. Then $\frac{d\psi}{dt}$ is the rate at which the invariable line describes its cone.

Since the body is turning about OI with an angular velocity ω, we have

$$OG\frac{d\psi}{dt} = \text{velocity of } G$$
$$= \omega \,.\, OG \sin GOI.$$

Also by Art. 209 we have

$$\frac{T}{G} = \omega \cos GOI.$$

Eliminating the angle GOI, we obtain

$$\left(\frac{d\psi}{dt}\right)^2 = \omega^2 - \frac{T^2}{G^2} \dots\dots\dots\dots\dots\dots (1).$$

Eliminating ω, we obtain

$$\frac{d\psi}{dt} = \frac{T}{G}\tan GOI \dots\dots\dots\dots\dots\dots (2).$$

Also $$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$$
$$= G^2\left\{\frac{\cos^2\alpha}{A^2} + \frac{\cos^2\beta}{B^2} + \frac{\cos^2\gamma}{C^2}\right\}.$$

Hence the formula

$$\left(\frac{d\psi}{dt}\right)^2 = G^2\left\{\frac{\cos^2\alpha}{A^2} + \frac{\cos^2\beta}{B^2} + \frac{\cos^2\gamma}{C^2}\right\} - \frac{T^2}{G^2} \dots\dots (3)$$

gives the velocity of OG in the terms of its direction cosines.

Any one of these three expressions will give the velocity with which the point fixed in space travels along the spheroconic on the surface of the ellipsoid of gyration. Other formulæ to find the motion will be given in a another article.

The Second and Third Cones.

215. *The Second Cone.* If a body having one point fixed be set in rotation about any axis, it will not continue to rotate about that axis, but the instantaneous axis will describe

a cone in the body. The equations to the instantaneous axis at any moment referred to the principal axes of the body are

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3},$$

and by Art. 204 we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = (A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2)\frac{T}{G^2};$$

therefore the equation to the cone is

$$Ax^2 + By^2 + Cz^2 = (A^2x^2 + B^2y^2 + C^2z^2)\frac{T}{G^2}.$$

This cone is of the second order, its axes coinciding with the principal diameters of the momental ellipsoid. It cuts the momental ellipsoid in a polhode and it becomes two planes when the initial conditions are such that $\frac{G^2}{T} = B$.

216. *The Third Cone.* The instantaneous axis also describes a cone in space. The equations of this cone cannot generally be found, but when they can be determined we have another geometrical representation of the motion. For suppose the two cones described by the instantaneous axis in the body and in space to be constructed. Since each of these cones contains two consecutive positions of their common generator, they will touch each other along the instantaneous axis. Then the points of contact having no velocity the motion will be represented by making the cone fixed in the body roll on the cone fixed in space. As those cones cut the momental ellipsoid in a polhode and a herpolhode they may be called respectively the *cone of the polhode* and the *cone of the herpolhode.*

217. *To find the velocity with which the instantaneous axis describes either of the two cones generated by it.*

First. *When the cones are given.*

Let the radii of curvature of the two cones at any point I of their line of contact OI be R, R' and let $OI = r$. These cones may be supposed to be made up of triangular planes, which become successively in contact at intervals of time dt. Let $d\epsilon$, $d\epsilon'$ be the angles between two successive planes in each cone. Then the body in turning round the instantaneous axis describes the angle $d\epsilon - d\epsilon'$. But it also describes ωdt;

$$\therefore \ \omega dt = d\epsilon - d\epsilon'.$$

Let $d\psi$ be the angle between the two positions of the instantaneous axis at the times t and $t + dt$. Then

$$d\epsilon = \frac{r d\psi}{R}, \quad d\epsilon' = \frac{r d\psi}{R'};$$

$$\therefore \ \omega = r \frac{d\psi}{dt} \cdot \left(\frac{1}{R} - \frac{1}{R'}\right).$$

If the point I be taken on the momental ellipsoid, we have $\omega = \frac{\sqrt{T}}{\epsilon^2} r$;

$$\therefore \ \frac{d\psi}{dt} = \frac{\sqrt{T}}{\epsilon^2} \cdot \left(\frac{1}{R} - \frac{1}{R'}\right)^{-1}.$$

Here R, R' are the radii of curvature of the two cones at the points where their line of contact cuts the momental ellipsoid.

218. Secondly. *To find the motion of the instantaneous axis along the cone which it describes in the body.*

If (x, y, z) be the co-ordinates of any point on the polhode referred to the principal axes of the body at the fixed point as co-ordinate axes, and r the radius vector, we have

$$\frac{\omega_1}{x} = \frac{\omega_2}{y} = \frac{\omega_3}{z} = \frac{\omega}{r} = \frac{\sqrt{T}}{\epsilon^2}.$$

Hence in any of the equations of Art. 204, we may replace angular velocities by the corresponding co-ordinates of the extremity of the instantaneous axis. We thus obtain

$$\left.\begin{aligned} x^2 &= \frac{BC}{(A-C)(A-B)}(-\lambda'_1+r^2) \\ y^2 &= \frac{CA}{(B-A)(B-C)}(-\lambda'_2+r^2) \\ z^2 &= \frac{AB}{(C-B)(C-A)}(-\lambda'_3+r^2) \end{aligned}\right\},$$

and
$$r\frac{dr}{dt} = \frac{\sqrt{T}}{\epsilon^2}\sqrt{(\lambda'_1-r^2)(\lambda'_2-r^2)(\lambda'_3-r^2)}$$

where $\lambda'_1 = \frac{T(B+C)-G^2}{BC} \cdot \frac{\epsilon^4}{T}$, with similar expressions for λ'_2 and λ'_3. Also we have the equations

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{\sqrt{T}}{\epsilon^2} \cdot \frac{B-C}{A} yz \\ \frac{dy}{dt} &= \frac{\sqrt{T}}{\epsilon^2} \cdot \frac{C-A}{B} zx \\ \frac{dz}{dt} &= \frac{\sqrt{T}}{\epsilon^2} \cdot \frac{A-B}{C} xy \end{aligned}\right\}.$$

From these equations we may find $\frac{ds}{dt}$ in terms of r.

219. Thirdly. *To find the motion of the instantaneous axis along the cone which it describes in space.*

Let, as before, (x, y, z) be the co-ordinates, referred to the principal axes of the body, of the point I in which the instantaneous axis cuts the polhode. During the motion of the body the polhode rolls on the herpolhode, it is therefore clear that the velocity of the extremity I of the instantaneous axis along the herpolhode is the same as that along the polhode. Hence the resolved parts of the velocity of I *in space* parallel to the axes are $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$. The moment of these about the axis of z is $x\dfrac{dy}{dt}-y\dfrac{dx}{dt}$. Hence the moment of these about the invariable line OG is

$$=\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)\frac{C\omega_3}{G}+\&c.+\&c.$$

Substituting for $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$ &c., ω_1 &c., their values in terms of x, y, z from the last article, the expression for the moment becomes

$$=\frac{T}{\epsilon^4}\cdot\frac{Cz^2}{G}\left(\frac{C-A}{B}x^2-\frac{B-C}{A}y^2\right)+\ldots\ldots$$

$$=\frac{T}{\epsilon^4}\cdot\frac{Cz^2}{G}\cdot\left\{\frac{C}{B-A}(-\lambda'_1+r^2)-\frac{C}{B-A}(-\lambda'_2+r^2)\right\}+\ldots\ldots$$

$$=\frac{T}{\epsilon^4}\cdot\frac{C^2(\lambda'_1-\lambda'_2)}{(A-B)G}\cdot z^2+\ldots\ldots.$$

Substituting again for z^2, &c. their values in terms of r^2, the expression for the moment becomes

$$=\frac{ABC^2(\lambda_1-\lambda_2)}{(A-B)(B-C)(C-A)G}\cdot(\lambda'_3-r^2)+\ldots\ldots$$

Now $(\lambda_1-\lambda_2)ABC=(TC-G^2)(A-B)$; therefore substituting, the coefficient of r^2 is easily seen to be $\dfrac{T}{G}$, and the constant part is

$$\frac{(TA-G^2)(TB-G^2)(TC-G^2)}{ABC\,.\,G^3}\cdot\frac{\epsilon^4}{T}-\frac{T^3}{G^3}\cdot\frac{\epsilon^4}{T}.$$

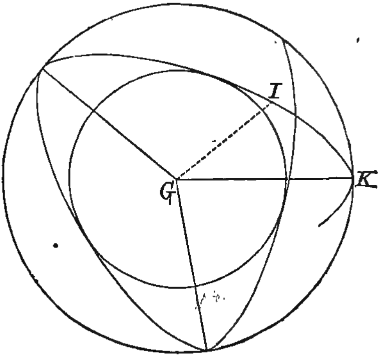
Let ρ be the distance of the extremity I of the instantaneous axis from the invariable line OG, and ϕ the angle the plane containing the invariable line and the instantaneous axis makes with any fixed plane passing through the invariable line. Then the moment of the velocity of I about

OG is $\rho^2 \dfrac{d\phi}{dt}$, and $r^2 = \rho^2 + \dfrac{T\epsilon^4}{G^2}$. Hence equating these two expressions for the moment, we have*

$$\rho^2 \frac{d\phi}{dt} = \frac{T}{G}\rho^2 + \frac{(TA - G^2)(TB - G^2)(TC - G^2)}{ABC\,.\,G^3} \cdot \frac{\epsilon^4}{T}.$$

If it be required to find $d\psi$ the angle between the two positions of the instantaneous axis at the times t and $t + dt$, we have the equation $\left(\dfrac{dr}{dt}\right)^2 + r^2\left(\dfrac{d\psi}{dt}\right)^2 = \left(\dfrac{d\rho}{dt}\right)^2 + \rho^2\left(\dfrac{d\phi}{dt}\right)^2$. The value of $\dfrac{dr}{dt}$ is given in Art. 218, and substituting for ρ and $\dfrac{d\phi}{dt}$ their values just found in terms of r, this equation gives $\dfrac{d\psi}{dt}$.

220. As the extremity I of the instantaneous axis describes the herpolhode on the invariable tangent plane, it alternately approaches to and recedes from G, the foot of the perpendicular. The least and greatest distances of I from G are found by making $\dfrac{d\rho}{dt} = 0$. Since $r^2 - \rho^2 = \text{constant}$ this gives $\dfrac{dr}{dt} = 0$. Hence by Art. 218 $r = \sqrt{\lambda'_3}$ or $\sqrt{\lambda'_1}$ or $\sqrt{\lambda'_2}$.



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* This is the result obtained by Poinsot in his *Theorie de la Rotation des Corps*. A simplification of the demonstration has been given by Mr Ferrers in *the Quarterly Journal of Mathematics*, 1864.

The three resulting values of ρ^2 are

$$-h^2\left(1-\frac{G^2}{AT}\right)\left(1-\frac{G^2}{BT}\right)$$

$$-h^2\left(1-\frac{G^2}{BT}\right)\left(1-\frac{G^2}{CT}\right)$$

$$-h^2\left(1-\frac{G^2}{CT}\right)\left(1-\frac{G^2}{AT}\right),$$

where $h^2=\frac{T\epsilon^4}{G^2}$ the length of the perpendicular from the fixed point on the invariable tangent plane. Since $\frac{G^2}{T}$ must lie between A and C, it will be seen that the last expression is the greatest of the three and is always positive. Of the other two the first or second is positive according as $\frac{G^2}{T}$ is greater or less than B. With these two real values of ρ as radii, describe circles about G the foot of the perpendicular on the invariable tangent plane. Then the herpolhode lies between these two circles. Put the greater radius $GK=k$, and let ϕ be the angle any radius vector GI makes with GK, then

$$\frac{d\phi}{dt}=\frac{T}{G}-\frac{TB-G^2}{BG}\cdot\left(\frac{k}{\rho}\right)^2,$$

and the angular velocity about OI is given by

$$\omega^2=\frac{T}{\epsilon^4}(\rho^2+h^2).$$

221. *To find the differential equation to the cone described in space by the instantaneous axis.*

Since the perpendicular on the tangent plane to the momental ellipsoid at the extremity of the instantaneous axis is constant, and $=\frac{\sqrt{T}}{G}\epsilon^2$, we have $r\,dr=\rho\,d\rho$. Hence the polar equation to the herpolhode is found by dividing the expressions for $r\dfrac{dr}{dt}$ and $\rho^2\dfrac{d\phi}{dt}$ given in Arts. 218 and 219, we thus get

$$\frac{d\rho}{\rho\, d\phi}=\frac{\dfrac{\sqrt{T}}{\epsilon^2}\sqrt{\left(\lambda_1'-\dfrac{T\epsilon^4}{G^2}-\rho^2\right)\left(\lambda_2'-\dfrac{T\epsilon^4}{G^2}-\rho^2\right)\left(\lambda_3'-\dfrac{T\epsilon^4}{G^2}-\rho^2\right)}}{\dfrac{T}{G}\rho^2+\dfrac{(TA-G^2)(TB-G^2)(TC-G^2)}{ABC\,.\,G^3}\dfrac{\epsilon^4}{T}}.$$

Let θ be the angle OI makes with OG, then the polar equation to the cone may be found by putting $\rho=\dfrac{\sqrt{T}}{G}\epsilon^2.\tan\theta$.

This equation cannot in general be integrated. If, however, the initial circumstances be such that $G^2=TB$, the integration can be effected.

The Fourth Cone.

222. Let O be the fixed point, OI the instantaneous axis. Let the angular velocity ω about OI be resolved into two, viz. a uniform angular velocity $\dfrac{T}{G}$ about the invariable line OG, and an angular velocity $\omega \sin IOG$ about a line OH lying in a plane fixed in space perpendicular to the invariable line, and passing through the fixed point O. Let this fixed plane be called the invariable plane at O. As the body moves, OH will describe a cone in the body which will always touch this fixed plane. The velocity of any point of the body lying for a moment in OH is unaffected by the rotation about OH, and the point has therefore only the motion due to the uniform angular velocity about OG. We have thus a new representation of the motion of the body. Let the cone described by OH in the body be constructed, and let it roll on the invariable plane at O with the proper angular velocity, while at the same time this plane turns round the invariable line with a uniform angular velocity $\dfrac{T}{G}$. The cone described by OH in the body is the fourth cone. It has been called by Poinsot the *Rolling and Sliding Cone.*

223. *To find a construction for the fourth cone.* Its generator OH is at right angles to OG, and lies in the plane IOG. Now OG is fixed in space; let OG' be the line in the body which, after an interval of time dt, will come into the position OG. Since the body is turning about OI, the plane GOG' is perpendicular to the plane GOI, and hence OH is perpendicular to both OG and OG'. That is, OH is perpendicular to the tangent plane to the cone described by OG in the body. The cone described by OH in the body is therefore the *reciprocal cone* of that described by OG.

The equation to the cone described by OG has been shown to be

$$\frac{AT-G^2}{A}x^2+\frac{BT-G^2}{B}y^2+\frac{CT-G^2}{C}z^2=0.$$

Hence the equation to the cone described by OH is

$$\frac{A}{AT-G^2}x^2+\frac{B}{BT-G^2}y^2+\frac{C}{CT-G^2}z^2=0.$$

The focal lines of the cone described by OH are perpendicular to the circular sections of the reciprocal cone, that is the cone described by OG. And these circular sections are the same as the circular sections of the ellipsoid of gyration. The equations to the focal lines of the cone described by OH are therefore

$$\left.\begin{array}{c}\dfrac{x^2}{\dfrac{1}{A}-\dfrac{1}{B}}=\dfrac{z^2}{\dfrac{1}{B}-\dfrac{1}{C}}\\ y=0\end{array}\right\}.$$

Thus the focal lines lie in the plane containing the axes of greatest and least moment, and are independent of the initial conditions.

This cone becomes a straight line in the case in which the cone described by OG becomes a plane, viz. when the initial conditions are such that $G^2=BT$.

224. *To find the motion of* OH *along the cone which it describes in the body.*

Since OG, OH, OI are always in the same plane, the angular motion of OH in space round the fixed line OG is the same as that of OI, and is given by the expression for $\frac{d\phi}{dt}$ in Art. 219. But the cone rolls on a plane which turns round OG with an angular velocity $\frac{T}{G}$. Hence the angular motion of OH relatively to this plane is $\frac{d\phi}{dt} - \frac{T}{G}$. But if $d\chi$ be the angle between two consecutive positions of OH in the body, this angular motion is expressed by $\frac{d\chi}{dt}$. Hence

$$\frac{d\chi}{dt} = \frac{(TA - G^2)(TB - G^2)(TC - G^2)}{ABCTG^3} \cdot \frac{\epsilon^4}{\rho^2}.$$

Here ρ is the distance from OG of the point I in which the instantaneous axis cuts the momental ellipsoid. If $OI = r$, we have $\rho = r \sin GOI = \frac{\epsilon^2}{\sqrt{T}} \cdot \omega \sin GOI$; and hence we have $\rho^2 = \frac{\epsilon^4}{T}\left(\omega^2 - \frac{T^2}{G^2}\right)$. The expression for the angular motion of OH is therefore

$$\frac{d\chi}{dt} = \frac{(TA - G^2)(TB - G^2)(TC - G^2)}{ABCG^3} \cdot \frac{1}{\omega^2 - \frac{T^2}{G^2}}.$$

Motion of the Principal Axes.

225. *To find the angular motions in space of the principal axes of the body.*

It will be most convenient to refer the motion to the invariable line OG as axis of z. Let (α, β, γ) be the angles the principal axes OA, OB, OC of the body make with the invariable line. Then we have

$$\cos\alpha = \frac{A\omega_1}{G}, \quad \cos\beta = \frac{B\omega_2}{G}, \quad \cos\gamma = \frac{C\omega_3}{G} \; \ldots\ldots \; (1).$$

Let (λ, μ, ν) be the angles the planes GOA, GOB, and GOC make with any fixed plane in space passing through OG. Then to find the motion of OA, we require $\frac{d\alpha}{dt}$ and $\frac{d\lambda}{dt}$.

By Euler's Equations, $A\frac{d\omega_1}{dt} - (B-C)\,\omega_2\omega_3 = 0.$

Substituting for ω_1, ω_2, ω_3 from equations (1) we get therefore

$$\sin\alpha\,\frac{d\alpha}{dt} = \left(\frac{1}{B} - \frac{1}{C}\right) G\cos\beta\cos\gamma \;\ldots\ldots\ldots\ldots(2).$$

But $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are connected by the equations

$$\left.\begin{aligned}\frac{\cos^2\alpha}{A} + \frac{\cos^2\beta}{B} + \frac{\cos^2\gamma}{C} &= \frac{T}{G^2}\\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma &= 1\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(3).$$

Hence

$$\left.\begin{aligned}\left(\frac{1}{B}-\frac{1}{C}\right)\cos^2\beta &= \frac{T}{G^2} - \frac{1}{C} - \left(\frac{1}{A}-\frac{1}{C}\right)\cos^2\alpha\\ \left(\frac{1}{C}-\frac{1}{B}\right)\cos^2\gamma &= \frac{T}{G^2} - \frac{1}{B} - \left(\frac{1}{A}-\frac{1}{B}\right)\cos^2\alpha\end{aligned}\right\}\ldots\ldots(4).$$

Substituting we find $\sin\alpha\frac{d\alpha}{dt}$

$$= G\sqrt{\left\{\frac{1}{C} - \frac{T}{G^2} - \left(\frac{1}{C}-\frac{1}{A}\right)\cos^2\alpha\right\}\left\{\frac{T}{G^2} - \frac{1}{B} + \left(\frac{1}{B}-\frac{1}{A}\right)\cos^2\alpha\right\}}$$
$$\ldots\ldots\ldots(5).$$

Again, let OI be the instantaneous axis, then the motion of A resolved perpendicular to GA is

$$\sin\alpha\,\frac{d\lambda}{dt} = \omega\sin AI.\cos GAI$$
$$= \omega\,.\,\frac{\cos GI - \cos\alpha\,.\,\cos AI}{\sin\alpha}.$$

But $\omega\cos GI$ is the resolved part of the angular velocity about OG which is equal to $\frac{T}{G}$, and $\omega\cos AI$ is the resolve part of the angular velocity about OA which is ω_1. Therefore

$$\sin^2\alpha\,\frac{d\lambda}{dt}=\frac{T}{G}-\omega_1\cos\alpha \qquad\text{(6)}$$

$$=\frac{T}{G}-\frac{G\cos^2\alpha}{A} \qquad\text{(7).}$$

226. In a similar manner we may find the motion of any other line OL fixed in the body and passing through the fixed point. Let (l, m, n) be the direction-cosines of OL referred to the principal axes at the fixed point. Let i be the angle OL makes with the invariable line, j the angle the plane LOG makes with any plane fixed in space and passing through OG. Then

$$\cos i=l\cos\alpha+m\cos\beta+n\cos\gamma \qquad\text{(8).}$$

Differentiating and proceeding as before, we have

$$\sin i\,\frac{di}{dt}=G\left\{\left(\frac{1}{B}-\frac{1}{C}\right)l\cos\beta\cos\gamma\right.$$
$$\left.+\left(\frac{1}{C}-\frac{1}{A}\right)m\cos\gamma\cos\alpha+\left(\frac{1}{A}-\frac{1}{B}\right)n\cos\alpha\cos\beta\right\},$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are to be eliminated by means of the three equations (3) and (8).

Also if ω' be the resolved part of the angular velocity about OL, we have by the same reasoning

$$\sin^2 i\,\frac{dj}{dt}=\frac{T}{G}-\omega'\cos i.$$

This may also be written in the form

$$\sin^2 i\,\frac{dj}{dt}=\frac{T}{G}-\left(\frac{l\cos\alpha}{A}+\frac{m\cos\beta}{B}+\frac{n\cos\gamma}{C}\right)G\cos i.$$

Let OL' be a perpendicular on the tangent plane at the point in which the line OL cuts the ellipsoid of gyration. Then the direction-cosines of OL' are $pr\frac{l}{A}$, $pr\frac{m}{B}$, $pr\frac{n}{C}$;

where p is the length of the perpendicular along OL' and r the length of the radius vector along OL. Let i' be the angle OL' makes with OG. Then

$$\sin^2 i \frac{dj}{dt} = \frac{T}{G} - \frac{G}{pr} \cos i \cos i'.$$

227. The equations (5) and (6) in Art. 225 for the determination of α and λ in terms of t cannot be completely integrated. They evidently depend on elliptic integrals. We may however observe that

(1) Since the left side of the equation (5) is necessarily real, we must have

$$\cos^2 \alpha < \frac{\frac{1}{C} - \frac{T}{G^2}}{\frac{1}{C} - \frac{1}{A}} \text{ and } > \frac{\frac{1}{B} - \frac{T}{G^2}}{\frac{1}{B} - \frac{1}{A}},$$

if the latter be positive. Thus the spiral described by any one A of the principal axes on the surface of a sphere whose centre is the fixed point lies between two concentric circles, which it alternately touches. If $\frac{1}{B} - \frac{T}{G^2}$ be negative, the polhode described by the instantaneous axis has its concavity turned towards the axis of C, and $\cos \alpha$ has no inferior limit. In this case the spiral always lies between two small circles on the sphere, one of which is exactly opposite the other.

(2) In order that $\frac{d\lambda}{dt}$ may vanish, we must have $\cos^2 \alpha = \frac{AT}{G^2}$, but this by substitution is found to make $\frac{d\alpha}{dt}$ imaginary. Thus $\frac{d\lambda}{dt}$ always keeps one sign. We may put the expression into the form

$$\frac{A}{G} \frac{d\lambda}{dt} = 1 + \frac{\frac{AT}{G^2} - 1}{1 - \cos^2 \alpha}.$$

If then the initial conditions are such that $\frac{G^2}{T}$ is less than the moment of inertia about the axis which describes the spiral we are considering, the angular velocity will be greatest when the axis is nearest the invariable line and least when the axis is furthest.

Motion when G² = BT.

228. The peculiarities of this case have been already alluded to several times. When the initial conditions are such that this relation holds between the Vis Viva and the Momentum of the body the whole discussion of the motion becomes more simple.

The fundamental equations of motion are

$$\left.\begin{aligned} A\omega_1^2 + B\omega_2^2 + C\omega_3^2 &= T \\ A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 &= G^2 \end{aligned}\right\} \text{..................} \quad (1).$$

Hence when $G^2 = BT$, we have

$$A(A-B)\,\omega_1^2 = C(B-C)\,\omega_3^2,$$

and

$$\left.\begin{aligned} \omega_1^2 &= \frac{B-C}{A-C}\cdot\frac{G^2 - B^2\omega_2^2}{AB} \\ \omega_3^2 &= \frac{A-B}{A-C}\cdot\frac{G^2 - B^2\omega_2^2}{BC} \end{aligned}\right\} \text{..............} \quad (2).$$

But

$$\frac{d\omega_2}{dt} = \frac{C-A}{B}\,\omega_1\omega_3;$$

$$\therefore\ \frac{d\omega_2}{dt} = \mp\sqrt{\frac{(A-B)(B-C)}{AC}}\cdot\frac{G^2 - B^2\omega_2^2}{B^2}.$$

When the initial values of ω_1 and ω_3 have like signs, $(C-A)\,\omega_1\omega_3$ is negative and therefore $\frac{d\omega_2}{dt}$ must be negative, hence in this expression the upper or lower sign is to be used according as the initial values of ω_1, ω_3 have like or unlike signs.

$$\therefore \frac{B}{G^2 - B^2\omega_2^{\,2}} \frac{d\omega_2}{dt} = \mp \frac{1}{B}\sqrt{\frac{(A-B)(B-C)}{AC}}$$

$$= \mp \frac{n}{B} \text{ suppose;}$$

$$\therefore \frac{G + B\omega_2}{G - B\omega_2} = E \,.\, e^{\mp \frac{2G}{B} nt},$$

where E is some undetermined constant,

$$\therefore \frac{B\omega_2}{G} = \frac{E \,.\, e^{\mp \frac{2G}{B} nt} - 1}{E \,.\, e^{\mp \frac{2G}{B} nt} + 1}.$$

As t increases indefinitely ω_2 approaches $\mp \frac{G}{B}$ as its limit and therefore by (2) ω_1 and ω_3 approach zero.

The conclusion is that the instantaneous axis ultimately approaches to coincidence with the mean axis of principal moment, but never actually coincides with it. It approaches the positive or negative end of the mean axis according as the initial values of ω_1, ω_3 have unlike or like signs.

229. *To find the motion of the body in space.* The cone which the invariable line OG describes in the body becomes one of the two planes

$$\sqrt{\frac{A-B}{A}} \,.\, x = \pm \sqrt{\frac{B-C}{C}}\, z.$$

Since $\frac{z}{x} = \frac{C\omega_3}{A\omega_1}$ the upper or lower sign is to be taken according as the initial values of ω_1, ω_3 have like or unlike signs. Both these planes pass through the mean axis, and are independent of the initial conditions except so far that $G^2 = BT$. These planes are the circular sections of the ellipsoid of gyration. Also since they are respectively perpendicular to the axes of the two right circular cylinders which envelope the momental ellipsoid, they are perpendicular to the asymptotes of the focal conic of the momental ellipsoid which lies in the plane of the greatest and least moments.

The motion is therefore such that a plane fixed in the body, viz. one of the circular sections of the ellipsoid of gyration, contains a line fixed in space, viz. the perpendicular to the plane of the impulsive couple. Since the body is brought from any position into the next by an angular velocity $\omega \cos IOG = \frac{T}{G}$ about OG, and an angular velocity $\omega \sin IOG$ about a perpendicular to OG, it follows that the plane fixed in the body turns round the line fixed in space with a uniform angular velocity equal to $\frac{T}{G}$ or $\frac{G}{B}$. At the same time the plane moves so that the line fixed in space appears to describe the plane with a variable* velocity. If β be the angle OG makes with the axis of mean moment this velocity will be $\frac{d\beta}{dt}$. We have

$$\cos \beta = \frac{B\omega_2}{G},$$

therefore by Art. 228

$$\frac{1 + \cos \beta}{1 - \cos \beta} = Ee^{\mp \frac{2G}{B} nt};$$

$$\therefore \cot \frac{\beta}{2} = \sqrt{E} \,.\, e^{\mp \frac{G}{B} nt},$$

and from this we easily find

$$\frac{d\beta}{dt} = \pm \frac{G}{B} n \,.\, \sin \beta,$$

which gives the required velocity.

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* This variable velocity is clearly $\omega \sin GOI = \frac{T}{G} \tan GOI$. But by the right-angled spherical triangle BIG in the figure of Art. 233, we have $\sin \beta = \cot B \tan GI$ and therefore $\frac{d\beta}{dt} = \frac{T}{G} \tan ED \sin \beta$.

230. Let the axis OB trace out an arc BB' on any sphere fixed in space whose centre is O, in the time dt. Then since BOG is the plane fixed in the body which turns about OG with an angular velocity $\frac{G}{B}$ we have $BGB' = \frac{G}{B}dt$. Let $B'm$ be an arc drawn perpendicular to GB. Then if δ be the angle GBB', we have

$$\cot\delta = \frac{Bm}{B'm} = \frac{d\beta}{\sin\beta \, . \, BGB'} = \pm n,$$

which is constant. Hence the path traced out by the axis of mean moment on a sphere whose centre is O, is a rhumb line which cuts all the great circles through G at an angle whose cotangent is $\pm n$. It will be seen that n is the tangent of the angle the separating polhode makes with the plane GOB fixed in the body.

Suppose ω_1, ω_3 to have initially like signs, then $\cot\delta = n$ and the positive side of the axis of mean moment will recede from the invariable line, and ultimately its negative side tends to coincidence with the invariable line. If ω_1, ω_3 have initially unlike signs, then $\cot\delta = -n$ and the positive side of the axis of mean moment ultimately approaches the invariable line.

231. The cone described by OH in the body is the reciprocal cone of that described by OG, and from it we may deduce reciprocal theorems. When $G^2 = BT$ this cone becomes one of two straight lines whose equations are

$$\left.\begin{array}{c} x\sqrt{\frac{1}{C} - \frac{1}{B}} = \pm z\sqrt{\frac{1}{B} - \frac{1}{A}} \\ y = 0 \end{array}\right\}.$$

These are perpendicular to the circular sections of the ellipsoid of gyration, and are the asymptotes of the focal conic of the momental ellipsoid. The motion is therefore such that a straight line fixed in the body describes a plane fixed in space, viz. the invariable plane. The straight line moves along this plane with a uniform angular velocity equal to $\frac{T}{G}$ or $\frac{G}{B}$.

And by the same reasoning as before (Art. 229) the angular velocity of the body about this line is $\pm \frac{G}{B} n \sin \beta$, where β is the angle the plane containing the axes of greatest and least moment of inertia makes with the invariable plane.

232. The cone which the instantaneous axis describes in the body also becomes one of the two planes

$$\sqrt{A(A-B)}\, x = \pm \sqrt{C(B-C)}\, z,$$

where, since $\frac{x}{\omega_1} = \frac{z}{\omega_3}$, the upper or lower sign is to be taken according as the initial values of ω_1, ω_3 have or have not the same sign. These planes are fixed in the body, being independent of initial conditions, except that $G^2 = BT$. They are perpendicular to the umbilical diameters of the ellipsoid of gyration, and are the diametral planes of the asymptotes of the focal conic in the momental ellipsoid.

233. The relations to each other of the several planes fixed in the body may be illustrated by the following figure. Let A, B, C be the points in which the principal axes of the body cut a sphere whose centre is O. Draw the planes BE, BD which are respectively described by the instantaneous axis OI and the invariable line OG; then by what precedes,

$$\tan CD = \sqrt{\frac{A}{C} \cdot \frac{B-C}{A-B}},$$

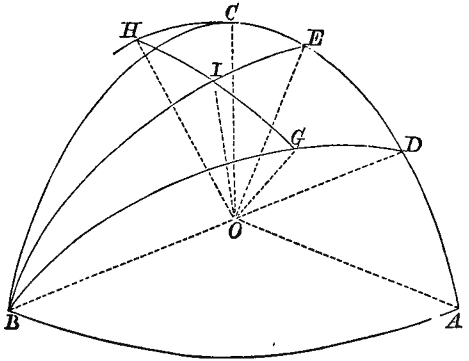
$$\tan CE = \sqrt{\frac{C}{A} \cdot \frac{B-C}{A-B}};$$

and therefore

$$\tan ED = \sqrt{\frac{(B-C)(A-B)}{AC}} = n.$$

Let OH be the fixed line in the body perpendicular to the plane OBD, so that H is the pole of the great circle BD. Then I is the intersection of HG and BE by Art. 222. Then, knowing by the previous investigation the value of

$BG = \beta$, the other angles may be found by Spherical Trigonometry.



Thus from the right-angled triangle BGI, we have

$$\cos B = \tan BG \,.\, \cot BI;$$

$$\therefore\ \tan BI = \tan \beta \,.\, \sqrt{1+n^2};$$

which determines the angular motion of the instantaneous axis along the separating polhode.

234. The instantaneous axis also describes a cone in space, whose equation may be found in the following manner:—

Let ϕ be the angle the plane IOG makes at the time t with some plane fixed in space passing through OG. Then, if in the time dt I has advanced to I', $d\phi = IGI'$. Since the body is turning about OI with an angular velocity ω, H which is fixed in the body has advanced to H' in the same time, where $HH' = \sin HI\omega dt$. But H', I', G are always in an arc of a great circle;

$$\therefore\ d\phi = \sin HI \,.\, \omega dt = \frac{G}{B}\,dt;$$

$$\therefore\ \phi = \frac{G}{B}\,t + F,$$

where F is some constant. Therefore the plane containing the instantaneous axis and the invariable line moves with uniform angular velocity round the invariable line.

Let θ be the angle the instantaneous axis makes with the invariable line. From the right-angled spherical triangle BGI,

$$\sin\beta = \tan\theta \,.\, \cot ED.$$

But $$\cot\frac{\beta}{2} = \sqrt{E} \,.\, e^{\mp\frac{G}{B}nt};$$

$$\therefore\ \frac{2n}{\tan\theta} = \frac{2}{\sin\beta} = \cot\frac{\beta}{2} + \tan\frac{\beta}{2}$$

$$= \sqrt{E} \,.\, e^{\mp n(\phi - F)} + \frac{1}{\sqrt{E}} e^{\pm n(\phi - F)}.$$

Let us choose the initial plane from which ϕ is measured so that $\sqrt{E} \,.\, e^{\mp nF} = 1$, then the equation to the cone traced out in space by the instantaneous axis is

$$\frac{2n}{\tan\theta} = e^{n\phi} + e^{-n\phi},$$

when $\phi = 0$, we have $\tan\theta = n$ or $\theta = DE$. Therefore the plane fixed in space from which ϕ is measured is the plane containing the axes of greatest and least moment at the time when that plane contains the invariable line.

Motion when A = B.

235. *To determine the motion of the body when two of the principal moments at the fixed point are equal.*

Let the body be set rotating with an angular velocity ω about an instantaneous axis OI, making an angle α with OC the axis of figure. Let OG be the perpendicular on the invariable plane.

The momental ellipsoid becomes in this case a spheroid, the axis of which is the axis of unequal moment of the body. From the symmetry of the figure it is evident that as the spheroid rolls on the invariable plane, the angles *GOC*, *GOI* are *constant*, and that the three axes *OI*, *OG*, *OC* are always in one plane.

By Art. 206 the direction-cosines of *OG* are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$. Hence if β be the angle *OG* makes with *OC*, we have

$$\cos\beta = \frac{C\omega_3}{\sqrt{A\omega_1^2 + B\omega_2^2 + C\omega_3^2}}.$$

But $\omega_1^2 + \omega_2^2 = \omega^2 \sin^2\alpha$ and $\omega_3^2 = \omega^2\cos^2\alpha$, hence

$$\cos\beta = \frac{C\cos\alpha}{\sqrt{A\sin^2\alpha + C\cos^2\alpha}}.$$

The angular velocity of the body about *OI* varies as the radius vector *OI* of the spheroid, and is therefore constant. Hence *OI* describes a right cone in the body round *OC* with a uniform angular velocity, and a right cone in space round *OG* with a uniform angular velocity.

The angular velocity ν of *OI* round *OC* in the body may be found most readily by referring to the original equations of motion in Art. 107. We have in this case

$$\left.\begin{aligned} \frac{d\omega_1}{dt} - n\frac{A-C}{A}\,\omega_2 &= 0 \\ \frac{d\omega_2}{dt} + n\frac{A-C}{A}\,\omega_1 &= 0 \end{aligned}\right\}.$$

Solving these in the usual way we have

$$\left.\begin{aligned} \omega_1 &= F\cos\left(\frac{A-C}{A}\,nt + f\right) \\ \omega_2 &= -F\sin\left(\frac{A-C}{A}\,nt + f\right) \end{aligned}\right\},$$

where F and f are arbitrary constants. Let χ be the angle the projection of the instantaneous axis on the plane perpendicular to *OC* makes with the fixed straight line which has been taken for the axis *OA*, then

$$\tan\chi = \frac{\omega_2}{\omega_1}, \quad \text{and } \nu = \frac{d\chi}{dt};$$

$$\therefore\ \chi = \frac{C-A}{A} nt + f;$$

$$\therefore\ \nu = \frac{C-A}{A} n,$$

where $n = \omega \cos \alpha$ is the angular velocity about the axis of figure.

The angular velocity ν' of OI round OG in space may be found from the consideration that OC, OI, OG are always in one plane. Describe a sphere round O as centre cutting OC, OL in C and L. The displacement CC' of C in the time dt due to the angular velocity ω round I is $\omega \sin \alpha dt$. Hence $\omega \frac{\sin \alpha}{\sin \beta} dt$ is the angle made by the two arcs CG, $C'G$ on the sphere. But, since OC, OI, OG are always in one plane, this is the angular velocity of OI about OG. Hence

$$\nu' = \omega \frac{\sin \alpha}{\sin \beta}$$

$$= \omega \frac{\sqrt{A^2 \sin^2 \alpha + C^2 \cos^2 \alpha}}{A}.$$

On correlated and Contrarelated Bodies.

236. *To compare the motions of different bodies acted on by the same initial couple.*

Let (α, β, γ) be the angles the principal axes OA, OB, OC of a body at the fixed point make with the invariable line OG. We have seen in Art. 225 that Euler's equations may be put into the form

$$\frac{d \cos \alpha}{dt} - G \left(\frac{1}{B} - \frac{1}{C} \right) \cos \beta \cos \gamma = 0 \ldots\ldots\ldots (1),$$

and two similar equations.

Let (λ, μ, ν) be the angles the planes GOA, GOB, GOC make with any plane fixed in space and passing through OG. Then

$$\sin^2\alpha \frac{d\lambda}{dt} = \frac{T}{G} - \frac{G\cos^2\alpha}{A}.$$

But as the vis viva of a body will be different from that of another body set in motion by the same couple, it will be convenient to substitute for T its value

$$\frac{T}{G^2} = \frac{\cos^2\alpha}{A} + \frac{\cos^2\beta}{B} + \frac{\cos^2\gamma}{C},$$

whence we have

$$\frac{d\lambda}{dt} = G\frac{\dfrac{\cos^2\beta}{B} + \dfrac{\cos^2\gamma}{C}}{\cos^2\beta + \cos^2\gamma} \quad\dots\dots\dots\dots\dots\dots (2),$$

and two similar equations.

If accented letters denote similar quantities for some other body, the equations of motion will be

$$\left.\begin{array}{r}\dfrac{d\cos\alpha'}{dt} - G'\left(\dfrac{1}{B'} - \dfrac{1}{C'}\right)\cos\beta'\cos\gamma' = 0\\ \&c. = 0\end{array}\right\}\dots\dots (3),$$

$$\left.\begin{array}{l}\dfrac{d\lambda'}{dt} = G'\dfrac{\dfrac{\cos^2\beta'}{B'} + \dfrac{\cos^2\gamma'}{C'}}{\cos^2\beta' + \cos^2\gamma'}\\ \&c. = \&c.\end{array}\right\}\dots\dots\dots\dots\dots\dots (4).$$

If then the bodies be such that

$$\frac{1}{A} - \frac{1}{A'} = \frac{1}{B} - \frac{1}{B'} = \frac{1}{C} - \frac{1}{C'} = \frac{1}{\Delta},$$

and be set in motion by equal impulsive couples $G' = G$, the equations (1) to find (α, β, γ) are the same as those (3) to find $(\alpha', \beta', \gamma')$. Therefore if these two bodies be initially placed with their principal axes parallel, and be set in motion by impulsive couples equal in magnitude and parallel in position, then after the lapse of any time t the principal axes of the two bodies will be still equally inclined to the axis of the couple. Also from equations (2) and (4), we have

$$\frac{d\lambda}{dt}-\frac{d\lambda'}{dt}=\frac{d\mu}{dt}-\frac{d\mu'}{dt}=\frac{d\nu}{dt}-\frac{d\nu'}{dt}=\frac{G}{\Delta},$$

so that the parallelism of the principal axes may be restored by turning the body, whose principal moments A', B', C' are the greater of the two, about the axis of the couple through an angle $\frac{G}{\lambda}t$ in the direction in which the impulsive couple acted.

The momental ellipsoids of these two bodies are clearly confocal, and the bodies themselves are said to be correlated.

Again, if the two bodies be such that

$$\frac{1}{A}+\frac{1}{A'}=\frac{1}{B}+\frac{1}{B'}=\frac{1}{C}+\frac{1}{C'}=\frac{1}{\Delta},$$

and be set in motion by equal and opposite impulsive couples $G'=-G$, the equations (1) and (3) are again identical, and by the same reasoning as before

$$\frac{d\lambda}{dt}-\frac{d\lambda'}{dt}=\&c.=\frac{G}{\Delta}.$$

Therefore if these two bodies be initially placed with their principal axes parallel and be set in motion by equal and opposite impulsive couples, then at the end of any time the principal axes will still be equally inclined to the axis of the couple, and the parallelism may be restored by turning either body through an angle $\frac{G}{\Delta}t$, in a direction opposite to that of the couple which acted on that body.

The momental ellipsoids of these bodies are contrafocal, *i.e.* have the *sum* of the squares of any two principal diameters the same in each ellipsoid. The bodies themselves are said to be contrarelated*.

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* In order that the angles which the principal axes make with the axis of the couple may be the same in each body, it is necessary that the cones described by the axis OG in the body should be the same. Hence by Art. 213, the two ellipsoids of gyration must have the same circular sections, or which

237. *To compare the angular velocities at any instant of two correlated or contrarelated bodies.*

Let ω be the angular velocity of one body at any instant, then following the usual notation we have

$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$$
$$= G^2 \left\{ \frac{\cos^2\alpha}{A^2} + \frac{\cos^2\beta}{B^2} + \frac{\cos^2\gamma}{C^2} \right\}.$$

If the same letters accented denote similar quantities for the other body, we have

$$\omega'^2 = G^2 \left\{ \frac{\cos^2\alpha}{A'^2} + \frac{\cos^2\beta}{B'^2} + \frac{\cos^2\gamma}{C'^2} \right\};$$

$$\therefore \omega'^2 - \omega^2 = G^2 \left\{ \cos^2\alpha \left(\frac{1}{A'^2} - \frac{1}{A^2} \right) + \cos^2\beta \left(\frac{1}{B'^2} - \frac{1}{B^2} \right) + \cos^2\gamma \left(\frac{1}{C'^2} - \frac{1}{C^2} \right) \right\}.$$

is the same thing, the two momental ellipsoids must have the same asymptotes to their hyperbolic focal conics. Also in order that the cones may be the same we must have

$$\frac{\frac{1}{A} - \frac{T}{G^2}}{\frac{1}{A'} - \frac{T'}{G'^2}} = \frac{\frac{1}{B} - \frac{T}{G^2}}{\frac{1}{B'} - \frac{T'}{G'^2}} = \frac{\frac{1}{C} - \frac{T}{G^2}}{\frac{1}{C'} - \frac{T'}{G'^2}}.$$

If we put each of these equal to some quantity r, we easily find

$$\frac{\frac{1}{A} - \frac{1}{B}}{\frac{1}{A'} - \frac{1}{B'}} = \frac{\frac{1}{B} - \frac{1}{C}}{\frac{1}{B'} - \frac{1}{C'}} = \frac{\frac{1}{C} - \frac{1}{A}}{\frac{1}{C'} - \frac{1}{A'}} = r.$$

If in the two bodies the angles between the principal axes and the axis of the couple are to be equal each to each at the *same time*, the equations of Art. 236 show that we must have in addition $\frac{G'}{G} = r$. In this case we have

$$\frac{d\lambda}{dt} - \frac{d\lambda'}{dt} = \frac{d\mu}{dt} - \frac{d\mu'}{dt} = \frac{d\nu}{dt} - \frac{d\nu'}{dt} = \frac{T}{G} - \frac{T'}{G'},$$

which is constant. The two cases considered in the text correspond to $r = \pm 1$.

If the bodies be correlated, this becomes

$$\omega'^2 - \omega^2 = \frac{G^2}{\Delta}\left\{\cos^2\alpha\left(\frac{1}{A'}+\frac{1}{A}\right) + \cos^2\beta\left(\frac{1}{B'}+\frac{1}{B}\right) + \cos^2\gamma\left(\frac{1}{C'}+\frac{1}{C}\right)\right\}$$

$$= \frac{1}{\Delta}(T'+T),$$

where T is the initial vis viva of one body and T' that of the other.

If the bodies be contrarelated, we may show in the same way that

$$\omega'^2 - \omega^2 = \frac{1}{\Delta}(T'-T).$$

In both cases the difference of the squares of the angular velocities at any instant is always the same.

We may also show that $T' \mp T = \frac{G^2}{\Delta}$, the upper or lower sign being taken according as the bodies are correlated or contrarelated.

238. When a body turns about a fixed point its motion in space is represented by making its momental ellipsoid roll on a fixed plane. This gives no representation of the *time* occupied by the body in passing from any position to any other. The preceding articles will enable us to supply this defect.

To give distinctness to our ideas let us suppose the momental ellipsoid to be rolling on a horizontal plane underneath the fixed point O, and that the instantaneous axis OI is describing a polhode about the axis of A. Let us now remove that half of the ellipsoid which is bounded by the plane of BC, and which does not touch the fixed plane. Let us replace this half by the half of another smaller ellipsoid which is confocal with the first. Let a plane be drawn parallel to the invariable plane to touch this ellipsoid in I' and suppose this plane also to be fixed in space. These two semi-ellipsoids may be considered as the momental ellipsoids of two correlated bodies. If they were not attached to each other and were free to move without interference, each would roll, the one on the fixed plane which touches at I, and the other on that which touches at I'. By what has been shown the upper ellipsoid (being the smallest) may be brought into parallelism with the lower by a rotation $\frac{G}{\Delta}t$ about the invariable line. If then the upper plane on which the upper ellipsoid rolls be made to turn round the invariable line as a fixed axis with an angular velocity $\frac{G}{\Delta}$, the two ellipsoids will always be in a state of parallelism, and may be supposed to be rigidly attached to each other.

Suppose then the upper tangent plane to be perfectly rough and capable of turning in a horizontal plane about a vertical axis which passes through the fixed point. As the nucleus is made to roll with the under part of its surface on the fixed plane below, the friction between the upper surface and the plane will cause the latter* to rotate about its axis. Then the time elapsed will be in a constant ratio to this motion of rotation, which may be measured off on an absolutely fixed dial face immediately over the rotating plane.

The preceding theory is taken from a memoir by Prof. Sylvester in the *Philosophical Transactions* for 1866. He proceeds to investigate in what cases the upper ellipsoid may be reduced to a disc. It appears that there are always two such discs and no more, except in the case of two of the principal moments being equal, when the solution becomes unique. Of these two discs one is correlated and the other contrarelated to the given body, and they will be respectively perpendicular to the axes of greatest and least moments of inertia.

Poinsot has shown that the motion of the body may be constructed by a cone fixed in the body rolling on a plane which turns uniformly round the invariable line. If, as in the preceding theory, we suppose the plane rough, and to be turned by the cone as it rolls on the plane, the angle turned through by the plane will measure the time elapsed.

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* As the ellipsoid rolls on the lower plane, a certain geometrical condition must be satisfied that the nucleus may not quit the upper plane or tend to force it upwards. This condition is that the plane containing OI, OI', must contain the invariable line, for then and then only the rotation about OI can be resolved into a component about OI' and a component about the invariable line. That this condition must be satisfied is clear from the reasoning in the text. But it is also clear from the known properties of confocal ellipsoids.

EXAMPLES.

1. A point P moves along a polhode traced on an ellipsoid, show that the length of the normal between P and any one of the principal planes at the centre is constant. Show also that the normal traces out on a principal plane a conic similar to the focal conic in that plane. Also the measure of curvature of an ellipsoid along any polhode is constant.

2. In the herpolhode $\dfrac{2mb}{r} = e^{m\theta} + e^{-m\theta}$, where

$$m^2 = \frac{(a^2 - b^2)(b^2 - c^2)}{b^2},$$

if the locus of the extremity of the polar subtangent of this curve be found and another curve be similarly generated from this locus, the curve thus obtained will be similar to the herpolhode.

3. If a right circular cone whose altitude a is double the radius of its base turn about its centre of gravity as a fixed point, and be originally set in motion about an axis inclined at an angle α to the axis of figure, the vertex of the cone will describe a circle whose radius is $\frac{3}{4} a \sin \alpha$.

4. A circular plate revolves about its centre of gravity as a fixed point. If an angular velocity ω were originally impressed on it about an axis making an angle α with its plane, a normal to the plane of the disc will make a revolution in space in time $\dfrac{2\pi}{\omega\sqrt{1 + 3\sin^2\alpha}}$.

5. If a solid of revolution be moving about its centre of gravity fixed, show that the plane containing the axis of figure and the instantaneous axis revolves uniformly about a line in itself and that these axes cannot be equally inclined to this line unless $C > 2A$, and in that case the inclination equals $\frac{1}{2}\cos^{-1}\frac{A}{C-A}$, C being the moment of inertia about the axis of figure, and A that about a line perpendicular to it.

6. A right cone the base of which is an ellipse is supported at G the centre of gravity, and has a motion communicated to it about an axis through G perpendicular to the line joining G, and the extremity B of the axis minor of the base, and in the plane through B and the axis of the cone. Determine the position of the invariable plane.

Result. The normal to the invariable plane lies in the plane passing through the axis of the cone and the axis of instantaneous rotation, and makes an angle $\tan^{-1}\frac{b}{16h}\cdot\frac{h^2+4a^2}{a^2+b^2}$.

7. A spheroid has a particle of mass m fastened at each extremity of the axis of revolution, and the centre of gravity is fixed. If the body be set rotating about any axis, show that the spheroid will roll on a fixed plane during the motion provided $\frac{m}{M}=\frac{1}{10}\left(1-\frac{a^2}{c^2}\right)$ where M is the mass of the spheroid, d and c are the axes of the generating ellipse, c being the axis of figure.

8. A lamina of any form rotating with an angular velocity α about an axis through its centre of gravity perpendicular to its plane has an angular velocity $\alpha\sqrt{\frac{B+C}{B-C}}$ impressed upon it about its principal axis of least moment, A, B, C being arranged in descending order of magnitude, show that at any time t the angular velocities about the principal axes are respectively $\frac{2\alpha}{e^{\alpha t}+e^{-\alpha t}}$, $-\sqrt{\frac{B+C}{B-C}}\,\alpha\,\frac{e^{\alpha t}-e^{-\alpha t}}{e^{\alpha t}+e^{-\alpha t}}$

and $\sqrt{\dfrac{B+C}{B-C}}\dfrac{2\alpha}{e^{\alpha t}+e^{-\alpha t}}$, and that it will ultimately revolve about the axis of mean moment.

9. A rigid body not acted on by any force is in motion about its centre of gravity: prove that if the instantaneous axis be at any moment situated in the plane of contact of either of the right circular cylinders described about the central ellipsoid, it will be so throughout the motion.

If a, b, c be the semi-axes of the central ellipsoid, arranged in descending order of magnitude, e_1, e_2, e_3 the eccentricities of its principal sections, Ω_1, Ω_2, Ω_3 the initial component angular velocities of the body about its principal axes, prove that the condition that the instantaneous axis should be situated in the plane above described is $\dfrac{\Omega_1}{e_1}=\dfrac{ab}{c^2}\dfrac{\Omega_3}{e_3}$.

10. A rigid lamina not acted on by any forces has one point fixed about which it can turn freely. It is started about a line in the plane of the lamina the moment of inertia about which is Q. Show that the ratio of the greatest to the least angular velocity is $\sqrt{A+Q}:\sqrt{B+Q}$ where A, B are the principal moments of inertia about axes in the plane of the lamina.

11. If the earth were a rigid body acted on by no force rotating about a diameter which is not a principal axis, show that the latitudes of places would vary and that the values would recur whenever $\sqrt{A-B}\sqrt{A-C}\int\omega_1 dt$ is a multiple to $2\pi\sqrt{BC}$. If a man were to lie down when his latitude is a minimum and to rise when it becomes a maximum, show that he would increase the vis viva and so cause a pole of the earth to travel from the axis of greatest moment of inertia towards that of least moment of inertia.

12. If a plane be drawn through the fixed point parallel to the invariable plane, prove that the area of the section of the momental ellipsoid cut off by this plane is constant throughout the motion.

13. If we draw three straight lines OA, OB, OC along the principal axes at the fixed point O of equal lengths, the sum of the areas conserved by these lines on the invariable plane is proportional to the time.

14. If the lengths OA, OB, OC be proportional to the radii of gyration about the axes respectively, the sum of the areas conserved by these lines on the invariable plane will also be proportional to the time.

15. The sum of the squares of the distances of the extremities of the principal diameters of the momental ellipsoid from the invariable line is constant throughout the motion.

16. If right lines are measured along the three principal axes of the body from the fixed point, and inversely proportional to the radii of gyration round those axes, the sum of the squares of the velocities of their extremities is constant throughout the motion.

17. A body moves about a fixed point O under the action of no forces. Show that if the surface $Ax^2 + By^2 + Cz^2 = (x^2 + y^2 + z^2)^2$ be traced in the body, the principal axes at O being the axes of co-ordinates, this surface throughout the motion will roll on a fixed sphere.

18. If $d\theta$ be the angle between two consecutive positions of the instantaneous axis, prove that

$$\omega^2 \left(\frac{d\theta}{dt}\right)^2 = \left(\frac{d\omega_1}{dt}\right)^2 + \left(\frac{d\omega_2}{dt}\right)^2 + \left(\frac{d\omega_3}{dt}\right)^2 - \left(\frac{d\omega}{dt}\right)^2.$$

19. The length of the spiral between two of its successive apsides, described in absolute space, on the surface of a fixed concentric sphere, by the instantaneous axis of rotation, is equal to a quadrant of the spherical ellipse described by the same axis on an equal sphere moving with the body. This is Booth's Theorem.

20. A body is set rotating with an initial angular velocity n about an axis which very nearly coincides with a principal axis OC at a fixed point O. The motion of the instantaneous axis in the body may be found by the following

formulæ. Let a sphere be described whose centre is O, and let I be the extremity of the radius vector which is the instantaneous axis at the time t. If (x, y) be the co-ordinates of the projection of I on the plane AOB referred to the principal axes OA, OB, then

$$x = \sqrt{B(B-C)}\, L \sin(pnt + M),$$

$$y = \sqrt{A(A-C)}\, L \cos(pnt + M),$$

where $p^2 = \dfrac{(B-C)(A-C)}{AB}$, and L, M are two arbitrary constants depending on the initial values of x, y.

21. If in the last question G be the point in which the sphere cuts the invariable line, and if (ρ, θ) be the spherical polar co-ordinates of C with regard to G as origin, then

$$\rho^2 = \frac{AB}{2G^2} L^2 \{2AB - C(A+B) + (A-B)\, C \cos 2(pnt + M)\},$$

$$\theta = \frac{G}{C} t + \frac{CT - G^2}{CG} \int \frac{dt}{\rho^2}.$$

22. A body which can turn freely about a fixed point at which two of the principal moments are equal and less than the third, is set in rotation about any axis. Owing to the resistance of the air and other causes, it is continually acted on by a retarding couple whose axis is the instantaneous axis of rotation and whose magnitude is proportional to the angular velocity. Show that the axis of rotation will continually tend to become coincident with the axis of unequal moment. In the case of the earth therefore, a near coincidence of the axis of rotation and axis of figure is not a proof that such coincidence has always held. Astronomical Notices, March 8, 1867.

23. If two bodies be so related that their ellipsoids of gyration are confocal, and be so initially placed that the angles (α, β, γ) $(\alpha', \beta', \gamma')$ their principal axes make with the invariable line of each are connected by the equations

$$\frac{\cos\alpha}{\sqrt{A}} = \frac{\cos\alpha'}{\sqrt{A'}}, \quad \frac{\cos\beta}{\sqrt{B}} = \frac{\cos\beta'}{\sqrt{B'}}, \quad \frac{\cos\gamma}{\sqrt{C}} = \frac{\cos\gamma'}{\sqrt{C'}},$$

and if these bodies be set in motion by two impulsive couples G, G' so that $G\sqrt{ABC} = G'\sqrt{A'B'C'}$; then the above relations will always hold between the angles (α, β, γ) $(\alpha', \beta', \gamma')$. Also $\frac{BC}{G}\frac{d\lambda}{dt} - \frac{B'C'}{G'}\frac{d\lambda'}{dt'}$ will be constant throughout the motion, where λ, λ' &c., are the angles the planes GOA, $G'O'A'$ make at the time t with their positions at the time $t = 0$.

24. An ellipsoid, whose centre O is fixed, contracts by cooling and being set in motion in any manner is under the action of no forces. Find the motion.

The principal diameters are principal axes at O throughout the motion. Let us take them as axes of reference. The expressions for the angular momenta about the axes are by Art. 118 $h_1' = A\omega_1$, $h_2' = B\omega_2$, $h_3' = C\omega_3$. The equations of Art. 120 then become

$$\left.\begin{aligned} \frac{d}{dt}(A\omega_1) - (B - C)\,\omega_2\omega_3 = 0 \\ \frac{d}{dt}(B\omega_2) - (C - A)\,\omega_3\omega_1 = 0 \\ \frac{d}{dt}(C\omega_3) - (A - B)\,\omega_1\omega_2 = 0 \end{aligned}\right\}.$$

Multiplying these equations by $A\omega_1$, $B\omega_2$, $C\omega_3$, adding and integrating we get $A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2$ equal to a constant throughout the motion. To obtain another integral, let $A = A_0 f(t)$, $B = B_0 f(t)$, $C = C_0 f(t)$ where $f(t)$ expresses the law of cooling which has been supposed such that the body changes its form very slowly. Let $\omega_1 f(t) = \Omega_1$, $\omega_2 f(t) = \Omega_2$, $\omega_3 f(t) = \Omega_3$, and put $\frac{dt'}{dt} = \frac{1}{f(t)}$, then the equations become

$$A_0\frac{d\Omega_1}{dt'} - (B_0 - C_0)\,\Omega_2\Omega_3 = 0,$$

and two similar equations. These may be treated as in Art. 204.

CHAPTER X.

EXAMPLES OF THE MOTION OF A BODY IN THREE DIMENSIONS.

239. In this Chapter it is proposed to discuss some cases of the motion of a rigid body in three dimensions as examples of the processes explained in Chapter V. The reader will find it an instructive exercise to attempt their solution by other methods; for example, the equations of Lagrange might be applied with advantage in some cases.

Motion of a Top.

240. *A body two of whose principal moments at the centre of gravity are equal moves about some fixed point* O *in the axis of unequal moment under the action of gravity. Determine the motion.* See Art. 167.

To give distinctness to our ideas we shall consider the body to be a top spinning on a perfectly rough horizontal plane.

Let the axis OZ be vertical. Let the axis of unequal moment at the centre of gravity be the axis OC and let this be called the axis of the body. Let h be the distance of the centre of gravity of the body from the fixed point O and let the mass of the body be taken as unity. Let OA, OB be any other two principal axes fixed in the body at right angles to OC and to each other.

If we take moments about the axis OC, we get by Euler's equations, (Art 107),

$$C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\,\omega_2=N \quad\text{.................}\quad (1).$$

But in our case $A=B$, and since the centre of gravity is in the axis OC, we have $N=0$. Hence ω_3 is constant and equal to its initial value. Let this be called n.

It is clear that the moment of the momentum about OZ will be constant throughout the motion. This principle gives

$$A\omega_1\cos ZA+B\omega_2\cos ZB+C\omega_3\cos ZC=E,$$

where E is some constant. Referring to the figure of Art. 109, we see that this is the same as

$$-A\omega_1\sin\theta\cos\phi+B\omega_2\sin\theta\sin\phi+C\omega_3\cos\theta=E.$$

Substituting for ω_1, ω_2 their values in terms of θ and ϕ from Art. 109, it will be found that when $B=A$, the angle ϕ disappears from the equation, and we have

$$A\sin^2\theta\frac{d\psi}{dt}+C\omega_3\cos\theta=E \quad\text{.................}\quad (2).$$

Again the equation of Vis Viva gives

$$A\omega_1^2+B\omega_2^2+C\omega_3^2=F-2gh\cos\theta,$$

where F is some constant. Substituting as before for ω_1, ω_2 we get, when $B=A$,

$$A\left(\frac{d\theta}{dt}\right)^2+A\sin^2\theta\left(\frac{d\psi}{dt}\right)^2+C\omega_3^2=F-2gh\cos\theta \quad\text{......}\quad (3),$$

where ϕ has again disappeared.

These two equations are sufficient to determine θ and ψ.

Eliminating $\frac{d\psi}{dt}$ between (2) and (3) we have

$$\left(\frac{d\theta}{dt}\right)^2+\left(\frac{E-Cn\cos\theta}{A\sin\theta}\right)^2=\frac{F-Cn^2-2gh\cos\theta}{A},$$

an equation to determine the changes of inclination of the axis to the vertical. From this equation we see that θ can never vanish unless $E = Cn$, for in any other case the left-hand side of this equation would become infinite. Hence the axis of the body cannot in general become vertical. Suppose the body to be set in motion in any way with its axis at an inclination i to the vertical. The axis will begin to approach or to fall away from the vertical according as the initial value of $\frac{d\theta}{dt}$ is negative or positive. This motion will continue until $\frac{d\theta}{dt}$ vanishes: it is evident from the equation that the axis will then begin to return, and will oscillate between two limiting angles. To find these limits we have the equation

$$(E - Cn\cos\theta)^2 - A(1 - \cos^2\theta)(F - Cn^2 - 2gh\cos\theta) = 0 \ldots (4).$$

This is a cubic equation to determine $\cos\theta$. It will be necessary to examine its roots. When $\cos\theta = -1$, the left-hand side is *positive;* when $\cos\theta = \cos i$, since the initial value of $\left(\frac{d\theta}{dt}\right)^2$ is essentially positive, the left-hand side is either zero or *negative:* hence the equation has one real root between $\cos\theta = -1$ and $\cos\theta = \cos i$. Again, the left-hand side is *positive* when $\cos\theta = +1$, and is *negative* when $\cos\theta = \infty$. Hence there is another real root between $\cos\theta = \cos i$ and $\cos\theta = 1$, and a third root greater than unity. This last root is inadmissible.

If the initial values of ω_1, ω_2 are zero, we have $E = Cn\cos i$, $F - Cn^2 = 2gh\cos i$. Hence the equation becomes

$$(\cos i - \cos\theta)^2 = \frac{2ghA}{C^2n^2} \cdot (1 - \cos^2\theta)(\cos i - \cos\theta).$$

Putting $\frac{C^2n^2}{2ghA} = 2p$, the roots of this equation are

$$\left.\begin{aligned} \cos\theta &= \cos i \\ \cos\theta &= p - \sqrt{1 - 2p\cos i + p^2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (5).$$

The value $\cos\theta = p + \sqrt{1 - 2p\cos i + p^2}$ is always greater than unity, for it is clearly decreased by putting unity for $\cos i$, and its value is then $p+1-p=1$. The body will therefore oscillate between the values of θ given by the equations (5).

If a top be set spinning on a *perfectly smooth horizontal plane*, its motion may be determined in the same way.

241. *A body, two of whose principal moments at the centre of gravity* G *are equal, turns about a fixed point* O *in the axis of unequal moment under the action of gravity. The axis* OG *being inclined to the vertical at an angle* α, *and revolving about it with a uniform angular velocity, find the condition that the motion may be steady, and the time of a small oscillation.*

The equations (2) and (3) of Art. 240 contain the solution of this problem. But if we use the equation of Vis Viva in the form (3) we shall have to take into account the squares of small quantities. It will be found more convenient to replace it by one of the equations of the second order from which it has been derived. The simplest method of obtaining this equation is to use Lagrange's Rule as in Art. 167. We thus obtain

$$A\frac{d^2\theta}{dt^2}-A\cos\theta\sin\theta\left(\frac{d\psi}{dt}\right)^2+Cn\sin\theta\frac{d\psi}{dt}=gh\sin\theta\ \dots\ (6).$$

This equation might also have been obtained by differentiating both (2) and (3), and eliminating $\dfrac{d^2\psi}{dt^2}$.

When the motion is steady, both θ and $\dfrac{d\psi}{dt}$ are constants. Let $\theta=\alpha$, $\dfrac{d\psi}{dt}=\mu$, then the equation (2) only determines the constant E, and (6) becomes

$$-A\cos\alpha\sin\alpha\mu^2+Cn\sin\alpha\mu=gh\sin\alpha\dots\dots\dots(7).$$

Rejecting the factor $\sin\alpha=0$, because α is *not* small, we have

$$n=\frac{A\cos\alpha\mu^2+gh}{C\mu}\ \dots\dots\dots\dots\dots\dots\dots(8).$$

This is the relation which must hold between the angular velocity of the body about its axis, and the angular velocity of that axis about the vertical when the motion is steady. But this relation is not necessary if the axis of the body be vertical.

Solving this equation we get

$$\mu = \frac{Cn \pm \sqrt{C^2n^2 - 4ghA\cos\alpha}}{2A\cos\alpha} \quad \text{..............(9)},$$

hence, in order that the motion may be steady, we must have n^2 not less than $\dfrac{4ghA\cos\alpha}{C^2}$.

When α and n are given, we can make the body move with either of these values of μ by giving the proper initial angular velocities to the body. By Art. 109, we see that these are

$$\left.\begin{aligned} \omega_1 &= -\mu\sin\alpha\cos\phi \\ \omega_2 &= \mu\sin\alpha\sin\phi \end{aligned}\right\}.$$

To find the small oscillation. Let $\theta = \alpha + \theta'$, and $\dfrac{d\psi}{dt} = \mu + \dfrac{d\psi'}{dt}$, where θ' and $\dfrac{d\psi'}{dt}$ are small quantities whose squares are to be neglected. Let α and μ be such that they contain the whole of the constant parts of θ and $\dfrac{d\psi}{dt}$, so that θ' and $\dfrac{d\psi'}{dt}$ contain only trigonometrical terms. Then when we substitute these values in equations (2) and (6), the constant parts must vanish of themselves. The equations thus obtained determine E and μ, and show that their values are the same as those determined when the motion is steady. The variable parts of the two equations become, after writing for Cn its value obtained from (8),

$$\left.\begin{aligned} &A\mu\sin\alpha\frac{d\psi'}{dt} - (gh - A\mu^2\cos\alpha)\,\theta' = 0 \\ &A\mu\frac{d^2\theta'}{dt^2} + \sin\alpha\,(gh - A\mu^2\cos\alpha)\frac{d\psi'}{dt} + \mu^3A\sin^2\alpha\theta' = 0 \end{aligned}\right\}.$$

To solve these, put

$$\theta' = F \sin (pt + f), \text{ and } \psi' = G \cos (pt + f).$$

Substituting, we have

$$\left.\begin{aligned} -A\mu \sin\alpha \,.\, pG &= (gh - A\mu^2 \cos\alpha)\, F \\ (A\mu p^2 - \mu^3 A \sin^2\alpha)\, F &= -(gh - A\mu^2 \cos\alpha) \sin\alpha \,.\, Gp \end{aligned}\right\}.$$

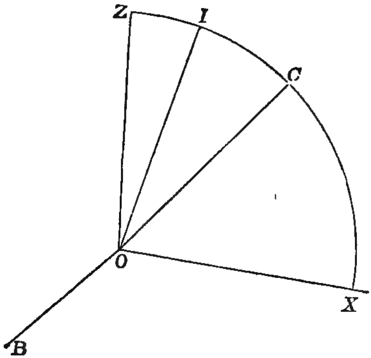
Multiplying these equations together, we have

$$p^2 = \frac{A^2\mu^4 - 2ghA \cos\alpha\, \mu^2 + g^2h^2}{A^2\mu^2},$$

and the required time is $\frac{2\pi}{p}$*. It is evident that p^2 is always positive, and therefore both the values of μ given by (9) correspond to stable motions.

It is to be observed that this investigation does not apply if α be very small, for in that case some of the terms rejected are of the same order of magnitude as those retained. A different mode of investigation is therefore required, this case will be considered in Art. 243.

242. We may also determine the *steady motion* very simply by another process. Let OC be the axis of the body,



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[*] This expression was given by the Rev. N. M. Ferrers of Gonville and Caius College, as the result of a problem proposed by him for solution in the Mathematical Tripos, 1859.

OI the instantaneous axis of rotation, OZ the vertical. Then when the motion is steady, these three must be in one vertical plane which revolves about OZ with a uniform angular velocity μ. Let Ω be the angular velocity about OI, then $\Omega \cos IC = n$. Let OB be the horizontal axis about which gravity tends to turn the body, then OB is perpendicular to the plane ZOC.

Since gravity generates an angular velocity $\frac{gh \sin \alpha}{A} dt$ in the time dt about OB, therefore by the parallelogram of angular velocities, the instantaneous axis OI has moved in the time dt through an angle $\frac{gh \sin \alpha}{A\Omega} dt$ in a plane perpendicular to the plane ZOI. Hence the angular velocity of I round Z due to the action of the forces is $\frac{d\psi_1}{dt} = \frac{gh \sin \alpha}{A\Omega} \cdot \frac{1}{\sin IZ}$.

Also, since the angular velocity of the body about OB is zero, the moments of the centrifugal forces* about the axes of OA, OC are zero. The moment about OB is $(A - C) n\Omega \sin IC\, dt$, and this generates an angular velocity $\frac{A-C}{A} n\Omega \sin IC\, dt$ about OB. Hence the angular velocity of I round Z due to the centrifugal forces of the body is

$$\frac{d\psi_2}{dt} = \frac{A - C}{A} n \frac{\sin IC}{\sin IZ},$$

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* It appears from Euler's Equations that the whole changes of ω_1, ω_2, ω_3 are not due merely to the direct action of the forces, but in part are due to the centrifugal force of the particles tending to carry them away from the axis about which they are revolving. For consider the equation

$$\frac{d\omega_3}{dt} = \frac{N}{C} + \frac{A-B}{C} \omega_1\omega_2.$$

Of the increase $d\omega_3$ in the time dt, the part $\frac{N}{C} dt$ is due to the direct action of the forces whose moment is N, and the part $\frac{A-B}{C} \omega_1\omega_2\, dt$ is due to the centrifugal force. This may be proved as follows.

If a body be rotating about an axis OI with an angular velocity ω, *then the moment of the centrifugal forces of the whole body about the axis Oz is* $(A - B)\, \omega_1\omega_2$.

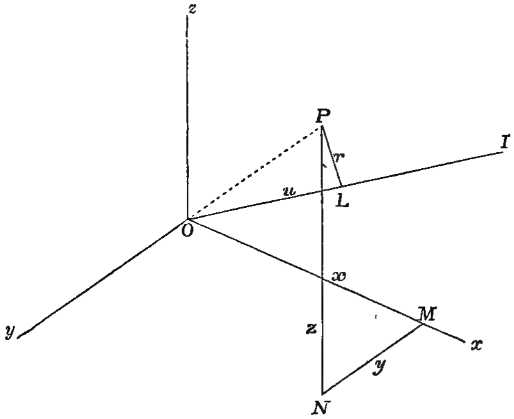
Let P be the position of any particle m and let x, y, z be its co-ordinates.

and the whole angular velocity is the sum of these two, i. e.

$$\mu = \left(\frac{gh\sin\alpha}{An}\cot IC + \frac{A-C}{A}n\right)\frac{\sin IC}{\sin IZ}$$

$$= \frac{gh\,.\sin\alpha\,.\cot IC + (A-C)n^2}{An\,(\sin\alpha\,.\cot IC - \cos\alpha)}.$$

Let PL be a perpendicular on OI, let $OL = u$, and $PL = r$. Then the centrifugal force of the particle m is $\omega^2 rm$ tending from OI.



The force $\omega^2 rm$ is evidently equivalent to the *four* forces $\omega^2 xm$, $\omega^2 ym$, $\omega^2 zm$, and $-\omega^2 um$ acting at P parallel to x, y, z, and u respectively.

$$\left.\begin{array}{l}\text{The moment of } \omega^2 xm \text{ round } Oz = -\,\omega^2 xym \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \omega^2 ym \ldots\ldots\ldots\ldots = \omega^2 xym \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \omega^2 zm \ldots\ldots\ldots\ldots = 0\end{array}\right\};$$

these three therefore produce no effect.

The force $-\omega^2 um$ parallel to OI is equivalent to the three, $-\omega\omega_1 um$, $-\omega\omega_2 um$, $-\omega\omega_3 um$, acting at P parallel to the axes, and their moment round Oz is evidently $\omega um\,(\omega_1 y - \omega_2 x)$. Now since the direction-cosines of OI are $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$, therefore by projecting the broken line x, y, z on OI, we get $u = \frac{\omega_1}{\omega}x + \frac{\omega_2}{\omega}y + \frac{\omega_3}{\omega}z$; therefore substituting for u, the moment of centrifugal forces about Oz is

$$= (\omega_1 y - \omega_2 x)\,(\omega_1 x + \omega_2 y + \omega_3 z)\,m,$$

$$= \omega_1{}^2 xy + \omega_1\omega_2 y^2 + \omega_1\omega_3 yz - \omega_1\omega_2 x^2 - \omega_2{}^2 xy - \omega_2\omega_3 xz\,.\,m.$$

Writing Σ before every term, and supposing the axes of x, y, z, to be

But when the motion is steady OZ, OI and OC are all in one plane. Now the angular velocity of C round I is Ω, and therefore its angular velocity round Z is $\mu = \Omega \frac{\sin IC}{\sin ZC}$. But $\Omega \cos IC = n$, hence, $\tan IC = \frac{\mu \sin \alpha}{n}$. Substituting this value of $\tan IC$ in the first value of μ, we get

$$\frac{gh}{\mu} = Cn - A\mu \cos \alpha,$$

the same expression as before.

243. *A body whose principal moments of inertia at the centre of gravity G are not necessarily equal has a point O in one of the principal axes at G fixed in space and moves about O under the action of gravity. Supposing the body to be per-*

principal axes, then the moment of the centrifugal forces about the principal axis $Oz = \omega_1\omega_2 \Sigma m (y^2 - x^2) = \omega_1\omega_2 (A - B)$.

Let the moments of the centrifugal forces about the principal axes of the body be represented by L, M, N so that

$$L = (B - C)\,\omega_2\omega_3, \quad M = (C - A)\,\omega_3\omega_1, \quad N = (A - B)\,\omega_1\omega_2,$$

and let G be their resultant couple. The couple G is usually called the *centrifugal couple.*

Since $L\omega_1 + M\omega_2 + N\omega_3 = 0$, it follows that the axis of the centrifugal couple is at right angles to the instantaneous axis.

Describe the momental ellipsoid at the fixed point O and let the instantaneous axis cut its surface in I. Let ON be a perpendicular from O on the tangent plane at I. The direction-cosines of ON are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$. Since $A\omega_1 L + B\omega_2 M + C\omega_3 N = 0$, it follows that the axis of the centrifugal couple is at right angles to the perpendicular ON.

The plane of the centrifugal couple is therefore the plane ION.

If Mk^2 be the moment of inertia of the body about the instantaneous axis of rotation we have $Mk^2 = \frac{\epsilon^4}{OI^2}$, and $T = Mk^2\omega^2$ is the Vis Viva of the body. We may then easily show that the magnitude G of the centrifugal couple is $G = T \tan \phi$, where ϕ is the angle ION.

This couple will generate an angular velocity of known magnitude about the diametral line of its plane. By compounding this with the existing angular velocity, the change in the position of the instantaneous axis might be found.

forming small oscillations about the position in which OG is vertical, find the motion.

Let OA, OB, OC, the principal axes at O be taken as the axes of reference, and let OC be that axis in which G lies. Let $OG = l$, and let G be *above* O. Let OV be a vertical line and let (p, q, r) be the direction-cosines of OV referred to OA, OB, OC. Then we have by Euler's Equations,

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 &= lgq \\ B\frac{d\omega_2}{dt} - (C - A)\,\omega_1\omega_3 &= -lgp \\ C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 &= 0 \end{aligned}\right\} \text{............ (1).}$$

Also p, q, r may be considered as the co-ordinates of a point in OV distant unity from O. Now this is a point fixed in space. Hence its velocities as given by Art. 111, are zero. We have therefore

$$\left.\begin{aligned} \frac{dp}{dt} - \omega_3 q + \omega_2 r &= 0 \\ \frac{dq}{dt} - \omega_1 r + \omega_3 p &= 0 \\ \frac{dr}{dt} - \omega_2 p + \omega_1 q &= 0 \end{aligned}\right\} \text{.................... (2).}$$

Now since OC remains always nearly vertical ω_1 and ω_2 are small quantities, we may therefore reject the product $\omega_1\omega_2$ in the last of equations (1). This gives ω_3 constant. Let this constant value be called n. For the same reason $r = 1$ nearly and p, q are both small quantities. Substituting we get the following linear equations,

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\,n\omega_2 &= lgq \\ B\frac{d\omega_2}{dt} - (C - A)\,n\omega_1 &= -lgp \end{aligned}\right\} \text{............ (3),}$$

$$\left.\begin{aligned}\frac{dp}{dt}&=qn-\omega_2\\ \frac{dq}{dt}&=-pn+\omega_1\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad(4).$$

To solve these, assume

$$\left.\begin{aligned}\omega_1&=F\sin(\lambda t+f)\\ \omega_2&=G\cos(\lambda t+f)\end{aligned}\right\},\qquad \left.\begin{aligned}p&=P\sin(\lambda t+f)\\ q&=Q\cos(\lambda t+f)\end{aligned}\right\},$$

substituting, we get

$$\left.\begin{aligned}A\lambda F-(B-C)\,nG&=glQ\\ B\lambda G-(A-C)\,nF&=glP\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad(5),$$

$$\left.\begin{aligned}\lambda P&=Qn-G\\ \lambda Q&=Pn-F\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ \ldots\ldots\ldots\quad(6).$$

Eliminating the ratios $F:G:P:Q$ we have

$$\lambda^2n^2(A+B-C)^2=\{gl+A\lambda^2+(B-C)\,n^2\}\,\{gl+B\lambda^2+(A-C)\,n^2\}.$$

If the values of λ thus found should be real, the body will make small oscillations about the position in which OG is vertical. If C be the greatest moment, and n^2 sufficiently great to make both $gl-(C-A)\,n^2$ and $gl-(C-B)\,n^2$ negative, then all the values of λ are real and the body will continue to spin with OG vertical. If G be beneath O, l is negative and it will be sufficient that OC should be the axis of greatest moment.

In order that the values of λ^2 may be real, we must have

$$\{gl\,(A+B)+n^2\,(AC+BC-2AB-C^2)\}^2$$
$$>4\,\{(B-C)\,n^2+gl\}\,\{(A-C)\,n^2+gl\}\,AB,$$

and in order that the two values of λ^2 may have the same sign we must have the last term of the quadratic positive;

$$\therefore\ \{(B-C)\,n^2+gl\}\,.\,\{(A-C)\,n^2+gl\}=\text{a positive quantity},$$

and in order that the values of λ^2 may be both positive, we must have the coefficient of λ^2 in the quadratic negative;

$$\therefore\ gl\,(A+B)<n^2\,(B-C)\,(A-C).$$

In the particular case in which $A = B$, both sides of the quadratic become a perfect square and we have

$$A\lambda^2 \pm (2A - C)\, n\lambda + (A - C)\, n^2 + gl = 0;$$

$$\therefore \lambda = \mp \frac{2A - C}{2A}\, n \pm \frac{\sqrt{C^2n^2 - 4Agl}}{2A}.$$

With the reservations mentioned in Art. 199, the necessary and sufficient condition of stability is in this case $n > \frac{2\sqrt{Agl}}{C}$. By referring to equations (5) and (6) it will be seen that when $A = B$ we have $F = G$ and $P = Q$. If λ_1, λ_2 be the two values of λ found above, we have

$$\left.\begin{aligned} p &= P_1 \sin(\lambda_1 t + f_1) + P_2 \sin(\lambda_2 t + f_2) \\ q &= P_1 \cos(\lambda_1 t + f_1) + P_2 \cos(\lambda_2 t + f_2) \end{aligned}\right\}.$$

Let θ be the angle OC makes with the vertical, then $r^2 = \cos^2\theta = 1 - \theta^2$, and hence*

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* In order to understand the relation which exists between these results and those of Art. 241, it will be necessary to determine the oscillations by some process which holds both when α is large and very small. This may be done as follows. We have by Vis Viva the equation (see Art. 240)

$$\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{E - Cn\cos\theta}{A\sin\theta}\right)^2 = \frac{F' - 2gh\cos\theta}{A},$$

where F' has been put for $F - Cn^2$. If we put $z = \cos\theta$, this takes the form

$$A^2\left(\frac{dz}{dt}\right)^2 + (E - Cnz)^2 = A\,(F' - 2ghz)\,(1 - z^2).$$

Let us assume as the solution of this equation

$$z = \cos\alpha + P\cos(\lambda t + f),$$

where P is so small that on substituting in the above equation we may neglect P^3. Substituting and equating to zero the coefficients of the several powers of $\cos(\lambda t + f)$ we get

$$\left.\begin{aligned} &A^2P^2\lambda^2 + (E - Cn\cos\alpha)^2 = A\,(F' - 2gh\cos\alpha)\,(1 - \cos^2\alpha) \\ &-(E - Cn\cos\alpha)\, Cn = -ghA - AF'\cos\alpha + 3ghA\cos^2\alpha \\ &-A^2\lambda^2 + C^2n^2 = -AF' + 6ghA\cos\alpha \end{aligned}\right\}.$$

Now let us change the constant E into another μ by putting

$$\frac{E - Cn\cos\alpha}{A\sin^2\alpha} = \mu + \gamma P^2,$$

$$\begin{aligned}\theta^2 &= p^2+q^2\\ &= P_1^2+P_2^2+2P_1P_2\cos\{(\lambda_1-\lambda_2)\,t+f_1-f_2\}.\end{aligned}$$

Also if, as in Art. 109, we let ϕ be the angle the plane containing OA, OC makes with the plane containing OC and the vertical OV, we have

$$\left.\begin{aligned} p &= -\sin\theta\cos\phi\\ q &= \sin\theta\sin\phi\end{aligned}\right\},$$

and hence

$$-\tan\phi=\frac{P_1\cos(\lambda_1t+f_1)+P_2\cos(\lambda_2t+f_2)}{P_1\sin(\lambda_1t+f_1)+P_2\sin(\lambda_2t+f_2)}.$$

Also since θ is very small we have

$$\psi=nt+\alpha-\phi,$$

where α is some constant, depending on the position of the arbitrary plane from which ψ is measured.

244. *A body whose principal moments of inertia are not necessarily equal has a point O fixed in space and moves about*

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where γ is to be so chosen as to remove the term $A^2P^2\lambda^2$ in our first equation. Since

$$\frac{d\psi}{dt}=\frac{E-Cn\cos\theta}{A\sin^2\theta},$$

we see that, when θ is not small, μ differs from the constant part of $\dfrac{d\psi}{dt}$ only by quantities depending on the *squares* of the small oscillation, and which are neglected in the text. Substituting for E and eliminating F' between the first and second equations we get

$$n=\frac{A\cos\alpha\mu^2+gh}{C\mu}.$$

Eliminating F' between the first and third and substituting for n we get

$$\lambda^2=\frac{\mu^4A^2-2ghA\cos\alpha\mu^2+g^2h^2}{A^2\mu^2}.$$

This process gives the period of the small oscillation in $\cos\theta$. When θ is finite this is the same as the oscillation in θ, since $\cos\theta=\cos\alpha-\sin\alpha\,\theta'$. When θ is very small, $\cos\theta=1-\dfrac{\theta^2}{2}$ and the time of oscillation in $\cos\theta$ is the same as that in θ^2. With this understanding it will be seen that there is a perfect agreement between the results of Arts. 243 and 241, when α is put equal to zero.

O under the action of gravity. It is required to find what cases of steady motion are possible in which one principal axis OC at O describes a right cone round the vertical while the angular velocity of the body round OC is constant; and to find the small oscillations.

Let OA, OB, OC the principal axes at O be, as before, taken as the axes of reference. Let (h, k, l) be the co-ordinates of the centre of gravity, and let the mass of the body be taken as unity. Let the rest of the notation be the same as before.

Then, by Euler's equations, we have

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\,\omega_2\omega_3 &= -g\,(kr - lq) \\ B\frac{d\omega_2}{dt} - (C-A)\,\omega_3\omega_1 &= -g\,(lp - hr) \\ C\frac{d\omega_3}{dt} - (A-B)\,\omega_1\omega_2 &= -g\,(hq - kp) \end{aligned}\right\} \text{........ I,}$$

and since p, q, r are the co-ordinates of a point vertically over O and therefore fixed in space, we have

$$\left.\begin{aligned} \frac{dp}{dt} &= \omega_3 q - \omega_2 r \\ \frac{dq}{dt} &= \omega_1 r - \omega_3 p \\ \frac{dr}{dt} &= \omega_2 p - \omega_1 q \end{aligned}\right\} \text{...................... II.}$$

Now, by the question, when the motion is steady, r and ω_3 are both constants. In this case the first two equations of (I) and (II) form a set of *linear* equations to find the four quantities p, q, ω_1, ω_2. The solution of these equations is therefore of the form

$$\left.\begin{aligned} \omega_1 &= F_0 + F_1 \sin(\lambda t + f) \\ \omega_2 &= G_0 + G_1 \cos(\lambda t + f) \end{aligned}\right\},$$

$$\left.\begin{aligned} p &= P_0 + P_1 \sin(\lambda t + f) \\ q &= Q_0 + Q_1 \cos(\lambda t + f) \end{aligned}\right\}.$$

But these must also satisfy the last of equations (I). Substituting we see that there will be a term on the left side of the form

$$-\tfrac{1}{2}(A-B)F_1G_1\sin 2(\lambda t+f).$$

But there will be no such term on the right side. Hence we must have either $A=B$, $F_1=0$ or $G_1=0$. The motion in the case in which $A=B$ has already been considered in Art. 241. Again, substituting in the last of equations (II) and equating to zero the coefficient of $\sin 2(\lambda t+f)$ we find

$$P_1G_1-F_1Q_1=0.$$

Substituting in the first two of equations (I) and equating to zero the coefficients of $\cos(\lambda t+f)$ and $\sin(\lambda t+f)$, we find

$$A\lambda F_1-(B-C)nG_1=glQ_1$$

$$-B\lambda G_1-(C-A)nF_1=-glP_1;$$

from these equations we have F_1, G_1, P_1, Q_1 all equal to zero and therefore ω_1, ω_2, p, q are all constant as well as the given constants ω_3 and r.

In this case the equations (II) give

$$\frac{\omega_1}{p}=\frac{\omega_2}{q}=\frac{\omega_3}{r},$$

so that the axis of revolution must be vertical. Let Ω be the angular velocity about the vertical. Then $\omega_1=p\Omega$, $\omega_2=q\Omega$, $\omega_3=r\Omega$. Substituting in equations (I) we get

$$\frac{h}{p}-\frac{A\Omega^2}{g}=\frac{k}{q}-\frac{B\Omega^2}{g}=\frac{l}{r}-\frac{C\Omega^2}{g}\;\ldots\ldots\ldots\ldots\text{(III)}.$$

Unless, therefore, two of the principal moments are equal, it is necessary for steady motion that the axis of rotation should be vertical and the centre of gravity (hkl) must lie in the vertical straight line whose equations are (III).

This straight line may be constructed geometrically in the following manner. Measure along the vertical a length $OV=\frac{g}{\Omega^2}$ and draw a plane through V perpendicular to OV to touch an ellipsoid confocal with an ellipsoid of gyration. The centre of gravity must lie in the normal at the point of contact.

To find the small oscillations about the steady motion, i.e. to determine whether this motion be stable or not, we must put

$$p = \cos\alpha + P_0 \sin\lambda t + P_1 \cos\lambda t,$$

with similar expressions for q, r, ω_1, ω_2, ω_3. Substituting we shall get twelve linear equations to determine eleven ratios. Eliminating these we have an equation to find λ. It is sufficient for stability that all the roots of this equation should be real.

Motion of a Sphere.

245. *To determine the motion of a sphere on any perfectly rough surface under the action of any forces whose resultant passes through the centre of the sphere.*

Let G be the centre of gravity of the body and let the moving axes GC, GA, GB be respectively a normal to the surface and some two lines at right angles to be afterwards chosen at our convenience. Let the motions of these axes be determined by the angular velocities θ_1, θ_2, θ_3 about their instantaneous positions in the manner explained in Art. 110. Let u, v, w be the velocities of G resolved parallel to the axes so that $w = 0$, and ω_1, ω_2, ω_3 the angular velocities of the body about these axes. Let F, F' be the resolved parts of the friction of the perfectly rough surface on the sphere parallel to the axes GA, GB, and let R be the normal reaction. Let X, Y, Z be the resolved parts of the impressed forces on the centre of gravity. Let k be the radius of gyration of the sphere about a diameter, a its radius. The equations of motion of the sphere are by Arts. 122 and 112,

$$\left.\begin{aligned} \frac{d\omega_1}{dt} - \theta_3\omega_2 + \theta_2\omega_3 &= \frac{F'a}{k^2} \\ \frac{d\omega_2}{dt} - \theta_1\omega_3 + \theta_3\omega_1 &= -\frac{Fa}{k^2} \\ \frac{d\omega_3}{dt} - \theta_2\omega_1 + \theta_1\omega_2 &= 0 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots \text{(1)},$$

$$\left.\begin{aligned} \frac{du}{dt} - \theta_3 v \qquad &= X + F \\ \frac{dv}{dt} + \theta_3 u \qquad &= Y + F' \\ -\theta_2 u + \theta_1 v &= Z + R \end{aligned}\right\} \quad \text{..............} \quad (2),$$

and since the point of contact of the sphere and surface is at rest, we have

$$\left.\begin{aligned} u - a\omega_2 &= 0 \\ v + a\omega_1 &= 0 \end{aligned}\right\} \quad \text{......................} \quad (3).$$

Eliminating F, F', ω_1, ω_2 from these equations, we get

$$\left.\begin{aligned} \frac{du}{dt} - \theta_3 v &= \frac{a^2}{a^2+k^2} X + \frac{k^2}{a^2+k^2} \theta_1 a\omega_3 \\ \frac{dv}{dt} + \theta_3 u &= \frac{a^2}{a^2+k^2} Y + \frac{k^2}{a^2+k^2} \theta_2 a\omega_3 \end{aligned}\right\} \quad \text{..............} \quad (4).$$

246. The meaning of these equations may be found as follows. They are the two equations of motion of the centre of gravity of the sphere, which we should have obtained if the given surface had been smooth and the centre of gravity had been acted on by accelerating forces $\frac{k^2}{a^2+k^2}\theta_1 a\omega_3$ and $\frac{k^2}{a^2+k^2}\theta_2 a\omega_3$ along the axes GA, GB, and by the same impressed forces as before reduced in the ratio $\frac{a^2}{a^2+k^2}$. The motion therefore of the centre of gravity in these two cases with the same initial conditions will be the same. More convenient expressions for these two additional forces may be found thus. The centre of gravity moves along a surface formed by producing all the normals to the given surface a constant length equal to the radius of the sphere. Let us take the axes GA, GB to be tangents to the lines of curvature of this surface and let ρ_1, ρ_2 be the radii of curvature of the normal sections through these tangents respectively. Then

$$\theta_1 = -\frac{v}{\rho_2}, \quad \theta_2 = \frac{u}{\rho_1} \quad \text{....................} \quad (5).$$

If G be the position of the centre of gravity at the time t, the quantity $\theta_3 dt$ is the angle between the projections of two successive positions of GA on the tangent plane at G. Let χ, χ' be the angles the radii of curvature of the lines of curvature at G make with the normal, then by geometrical considerations we shall have

$$\theta_3 = \frac{u}{\rho_1} \tan \chi + \frac{v}{\rho_2} \tan \chi' \dots\dots\dots\dots\dots\dots (6).$$

We have also an expression for ω_3 given by equations (1). Substituting for ω_1, ω_2 from the geometrical equations (3) we get

$$\frac{d\omega_3}{dt} = \frac{uv}{a} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) \dots\dots\dots\dots\dots\dots (7).$$

The solution of the equations may be conducted as follows. Let (x, y, z) be the co-ordinates of the centre of the sphere. Then u, v may be found from the equation to the surface in terms of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ by resolving parallel to the axes of reference. If we eliminate u, v, θ_1, θ_2, θ_3 by means of (4), (5), and (6), we shall get three equations containing x, y, z, ω_3, and their differential coefficients with respect to t. These together with the equation to the surface will be sufficient to determine the motion at any time. One integral can always be found by the principle of Vis Viva. Since the sphere is turning about the point of contact as an instantaneously fixed point we have

$$(a^2 + k^2)(\omega_1^2 + \omega_2^2) + k^2\omega_3^2 = 2\phi,$$

where ϕ is the force function of the impressed forces. This is the same as

$$u^2 + v^2 + \frac{k^2a^2}{a^2 + k^2}\omega_3^2 = 2\frac{a^2}{a^2 + k^2}\phi \dots\dots\dots (8),$$

and the right-hand side of this equation is twice the force function of the altered impressed forces.

247. If the given surface on which the sphere rolls be a plane, we have ρ_1 and ρ_2 both infinite, hence θ_1, θ_2 are both

zero. If therefore a homogeneous sphere roll on a perfectly rough plane under the action of any forces whatever of which the resultant passes through the centre of the sphere, the motion of the centre of gravity is the same as if the plane were smooth, and all the forces were reduced in ratio $\frac{5}{7}$ to 1. And it is also clear that the plane is the only surface which possesses this property.

248. If the given surface on which the sphere rolls be another sphere of radius $b - a$, we have $\rho_1 = \rho_2 = b$. Hence ω_3 is constant; let this constant value be called n, and let V be the velocity of the centre of gravity. Since every normal section is a principal section, we may take GA a tangent to the path. Hence the motion of the centre of gravity is the same as if the whole mass collected at that point were acted on by an accelerating force $\frac{k^2}{a^2+k^2}\frac{anV}{b}$ in a direction perpendicular to the path, and all the impressed forces were reduced in the ratio $\frac{a^2}{a^2+k^2} : 1$. Since this additional force acts perpendicular to the path, it will not appear in the equation of Vis Viva. Hence the velocity of the centre of gravity in any position is the same as if it had arrived there simply under the action of the reduced forces. Let O be the centre of the fixed sphere, θ the angle OG makes with the vertical OZ, and ψ the angle the plane ZOG makes with any fixed plane passing through OG. Then by Vis Viva we have

$$\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\psi}{dt}\right)^2 = E - \frac{2g}{b}\cos\theta,$$

where E is some constant to be determined from the initial conditions. This also follows from equation (8).

Also taking moments about OZ, we have

$$\frac{b}{\sin\theta}\frac{d}{dt}\left(\sin^2\theta\frac{d\psi}{dt}\right) = \frac{k^2}{a^2+k^2}an\frac{d\theta}{dt},$$

an equation which will be found to be a transformation of the second of equations (4).

Integrating this equation we have

$$\sin^2\theta\,\frac{d\psi}{dt}=E'-\frac{k^2}{a^2+k^2}\,\frac{an}{b}\cos\theta,$$

where E' is some constant.

These two equations will suffice to determine $\frac{d\theta}{dt}$ and $\frac{d\psi}{dt}$ under any given initial conditions.

249. Let the given rough surface be any surface of revolution placed with its axis of figure vertical and vertex upwards, and let gravity be the only impressed force. In this case the meridians and parallels are the lines of curvature. Let the axis of figure be the axis of Z. Let θ be the angle the axis GC makes with the axis of Z, ψ the angle the plane containing Z and GC makes with any fixed vertical plane.

Then $\quad \theta_1=-\sin\theta\,\frac{d\psi}{dt},\quad \theta_2=\frac{d\theta}{dt},\quad \theta_3=\cos\theta\,\frac{d\psi}{dt}.$

Hence the equations (4) become

$$\frac{du}{dt}-\cos\theta\,\frac{d\psi}{dt}v=\frac{a^2}{a^2+k^2}g\sin\theta-\frac{k^2}{a^2+k^2}a\omega_3\sin\theta\,\frac{d\psi}{dt}\;\ldots\ldots\text{(i)},$$

$$\frac{dv}{dt}+\cos\theta\,\frac{d\psi}{dt}u=\frac{k^2}{a^2+k^2}a\omega_3\frac{d\theta}{dt}\;\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and equation (8) becomes

$$u^2+v^2+\frac{k^2}{a^2+k^2}a^2\omega_3{}^2=E+2g\,\frac{a^2}{a^2+k^2}\int\rho\sin\theta\,d\theta\ldots\text{(iii)},$$

where E is some constant, and ρ is the radius of curvature of the meridian. Also we have by (7)

$$\frac{d\omega_3}{dt}=-\frac{uv}{a}\left(\frac{1}{\rho}-\frac{\sin\theta}{r}\right)\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)},$$

where r is the distance of the centre of the sphere from the axis of z. The geometrical equations (5) become

$$u=\rho\,\frac{d\theta}{dt},\quad v=r\,\frac{d\psi}{dt}\;\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}.$$

To solve these, we may put (ii) into the form

$$\frac{dv}{d\theta}+\cos\theta\frac{d\psi}{d\theta}u=\frac{k^2}{a^2+k^2}a\omega_3,$$

which by (v) becomes

$$\frac{dv}{d\theta}+\frac{\rho\cos\theta}{r}v=\frac{k^2}{a^2+k^2}a\omega_3;$$

differentiating this, we have by (iv),

$$\frac{d^2v}{d\theta^2}+\frac{\rho\cos\theta}{r}\frac{dv}{d\theta}+Pv=0\text{ ....(vi)},$$

where $$P=\frac{d}{d\theta}\left(\frac{\rho\cos\theta}{r}\right)+\frac{k^2}{k^2+a^2}\left(1-\frac{\rho\sin\theta}{r}\right).$$

Now ρ and r may be found from the equation to the meridian curve as functions of θ. Hence P is a known function of θ. Solving this linear equation we have v found as a function of θ. Then by (iv) we have

$$\frac{d\omega_3}{d\theta}=-\frac{v}{a}\left(1-\frac{\rho\sin\theta}{r}\right),$$

and thence having found ω_3 we have u by equation (iii). Knowing u and v; θ and ψ may be found by equations (v).

250. *A heavy sphere is placed in equilibrium on the highest point of a surface of any form and being slightly disturbed makes small oscillations, find the motion.*

Let O be the highest point of the surface on which the centre of gravity G always lies. Let the tangents to the lines of curvature at O be taken as the axes of x and y, and let (x, y, z) be the co-ordinates of G. We shall assume that O is not a singular point on the surface. In order to simplify the general equations of motions (4) we shall take as the axes GA and GB the tangents to the lines of curvature at G. But since G always remains very near O, the tangents to the lines of curvature at G will be nearly parallel to those at O. So that to the first order of small quantities we have

$$\theta_1 = -\frac{1}{\rho_2}\frac{dy}{dt}, \quad \theta_2 = \frac{1}{\rho_1}\frac{dx}{dt},$$

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt},$$

and θ_3 will be a small quantity of at least the first order. Also since the sphere is supposed not to deviate far from the highest point of the surface, we have ω_3 constant, let this constant be called n.

The equation to the surface on which G moves, in the neighbourhood of the highest point, is

$$z = -\frac{1}{2}\left(\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2}\right).$$

The equation to the normal at x, y, z is

$$\frac{\xi - x}{-\dfrac{x}{\rho_1}} = \frac{\eta - y}{-\dfrac{y}{\rho_2}} = \frac{\zeta - z}{-1}.$$

Hence the resolved parts parallel to the axes of the normal pressure R on the sphere are $R\dfrac{x}{\rho_1}$, $R\dfrac{y}{\rho_2}$ and R.

The equations of motion (4) therefore become

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{a^2}{a^2+k^2} R\frac{x}{\rho_1} - \frac{k^2}{a^2+k^2}\frac{dy}{dt}\frac{an}{\rho_2} \\ \frac{d^2y}{dt^2} &= \frac{a^2}{a^2+k^2} R\frac{y}{\rho_2} + \frac{k^2}{a^2+k^2}\frac{dx}{dt}\frac{an}{\rho_1} \\ \frac{d^2z}{dt^2} &= R - g \end{aligned}\right\}.$$

But z is a small quantity of the second order, hence the last equation gives $R = g$.

To solve these equations, we put

$$\left.\begin{aligned} x &= F\cos(\lambda t + f) \\ y &= G\sin(\lambda t + f) \end{aligned}\right\}.$$

These give

$$\left.\begin{aligned}\left(\lambda^2 + \frac{a^2}{a^2+k^2}\frac{g}{\rho_1}\right)F &= \frac{k^2}{a^2+k^2}\frac{a\lambda n}{\rho_2}\,G\\ \left(\lambda^2 + \frac{a^2}{a^2+k^2}\frac{g}{\rho_2}\right)G &= \frac{k^2}{a^2+k^2}\frac{a\lambda n}{\rho_1}\,F\end{aligned}\right\}.$$

The equation to find λ is therefore

$$\left(\lambda^2 + \frac{a^2}{a^4+k^2}\frac{g}{\rho_1}\right)\left(\lambda^2 + \frac{a^2}{a^2+k^2}\frac{g}{\rho_2}\right) = \frac{k^4}{(a^2+k^2)^2}\frac{a^2\lambda^2 n^2}{\rho_1\rho_2}.$$

This is a quadratic equation to determine λ^2. In order that the motion may be oscillatory it is necessary and sufficient that the roots should be both positive. If ρ_1, ρ_2 be both negative, so that the sphere is placed like a ball inside a cup, the roots of the quadratic are positive for all values of n. If ρ_1, ρ_2 have opposite signs the roots cannot be both positive. If ρ_1, ρ_2 be both positive the two conditions of stability will be found to reduce to

$$n^2 > \frac{a^2+k^2}{k^4}\,g\,(\sqrt{\rho_1}+\sqrt{\rho_2})^2.$$

If ρ_1 be infinite, it is necessary that ρ_2 should be negative, and in that case the two values of λ^2 are $-\dfrac{a^2}{a^2+k^2}\dfrac{g}{\rho_2}$ and zero, which are both independent of n. If $\rho_1 = \rho_2$, we have $F = G$. In this case if θ be the inclination of the normal to the vertical, we have $\theta^2 = \dfrac{x^2+y^2}{\rho^2}$ and, as in Art. 243, we find

$$\theta^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\{(\lambda_1-\lambda_2)\,t + f_1 - f_2\},$$

where λ_1, λ_2 are the roots of the quadratic

$$\lambda^2 \pm \frac{k^2}{a^2+k^2}\frac{an}{\rho}\lambda + \frac{a^2}{a^2+k^2}\frac{g}{\rho} = 0.$$

251. *A perfectly rough surface of revolution is placed with its axis vertical. Determine the circumstances of motion that a heavy sphere may roll on it so that its centre describes a*

horizontal circle. And this state of steady motion being disturbed, find the small oscillations.

In this case we must recur to the equations of Art. 249, and let us adopt the notation of that article, except that to shorten the expressions we shall put for k^2 its value $\frac{2}{5}a^2$.

To find the steady motion. We must put $u, v, \omega_3, \theta, \frac{d\psi}{dt}$ all constant. Let α, μ and n be the constant values of θ, $\frac{d\psi}{dt}$ and ω_3. Then we have $u = 0$, $v = b\mu$, where b is the constant value of r.

The equation (i) becomes

$$-b\cos\alpha\,\mu^2 = \frac{5}{7}g\sin\alpha - \frac{2}{7}an\sin\alpha\mu\,.$$

The other dynamical equations are satisfied without giving any relation between the constants.

If the motion be steady, we have therefore

$$n = \frac{5}{2}\frac{g}{a\mu} + \frac{7}{2}\frac{b}{a}\mu\cot\alpha\,;$$

thus for the same value of n we have two values of μ, which correspond to different initial values of v.

We have the geometrical relation $\omega_1 = -\frac{v}{a}$, so that ω_1 and n have opposite signs. Hence the axis of rotation which necessarily passes through the point of contact of the sphere and the rough surface makes an angle with the vertical less than that made by the normal at the point of contact.

By inspecting the expression for n, it will be seen that it is a minimum when

$$\frac{5}{2}\frac{g}{a\mu} = \frac{7}{2}\frac{b\mu}{a}\cot\alpha,$$

and therefore $n^2 = 35\frac{bg}{a^2}\cot\alpha\,,\quad \mu^2 = \frac{5}{7}\frac{g}{b}\tan\alpha.$

When $\alpha=0$, we have $b=0$ also, but if c be the radius of curvature at the vertex of the surface on which G always lies, we have $b=c\sin\alpha$, and the expressions become

$$n^2=35\frac{cg}{a^2},\quad \mu^2=\frac{5}{7}\frac{g}{c}.$$

If the vertex of the surface of revolution be downwards, so that the sphere rolls on the inside, we have only to change the signs of a and g.

To find the small oscillation.

Put $\theta=\alpha+\theta'$, $\frac{d\psi}{dt}=\mu+\frac{d\psi'}{dt}$, where α and μ are supposed to contain *all* the constant parts of θ and $\frac{d\psi}{dt}$, so that θ' and $\frac{d\psi'}{dt}$ only contain trigonometrical terms. Let $c-a$ be the radius of curvature of the surface of revolution at the point of contact of the sphere in steady motion, so that ρ differs from c only by small quantities, and may be put equal to c in the small terms. Also we have $r=b+c\cos\alpha\,.\,\theta'$.

Now by equations (iv) and (v) of Art. 249 we have

$$\begin{aligned}\frac{d\omega_3}{dt}&=\frac{d\theta}{dt}\frac{d\psi}{dt}\frac{\rho\sin\theta-r}{a}\\&=\frac{d\theta'}{dt}\mu\frac{c\sin\alpha-b}{a};\end{aligned}$$

$$\therefore\ \omega_3=\mu\frac{c\sin\alpha-b}{a}\theta'+n,$$

where n is the whole of the constant part of ω_3.

Again, from equation (ii), we have

$$-\frac{1}{a}\frac{d}{dt}\left(r\frac{d\psi}{dt}\right)-\frac{\rho}{a}\frac{d\theta}{dt}\cos\theta\frac{d\psi}{dt}+\frac{k^2}{a^2+k^2}\omega_3\frac{d\theta}{dt}=0;$$

$$\therefore\ -\frac{\mu}{a}c\cos\alpha\frac{d\theta'}{dt}-\frac{b}{a}\frac{d^2\psi'}{dt^2}-\frac{c\cos\alpha\,\mu}{a}\frac{d\theta'}{dt}+\frac{2}{7}n\frac{d\theta'}{dt}=0;$$

integrating we have

$$\left(\frac{2}{7}n - \frac{2\mu c\cos\alpha}{a}\right)\theta' = \frac{b}{a}\frac{d\psi'}{dt},$$

the constant being put zero because θ' and ψ' only contain trigonometrical terms.

Thirdly, from equation (i), we have

$$\frac{1}{a}\frac{d}{dt}\left(\rho\frac{d\theta}{dt}\right) - \frac{r}{a}\left(\frac{d\psi}{dt}\right)^2\cos\theta + \frac{2}{7}\omega_3\sin\theta\frac{d\psi}{dt} = \frac{5}{7}\frac{g}{a}\sin\theta;$$

$$\therefore\ \frac{c}{a}\frac{d^2\theta'}{dt^2} - \frac{b+c\cos\alpha\theta'}{a}(\cos\alpha - \sin\alpha\theta')\left(\mu^2 + 2\mu\,.\,\frac{d\psi'}{dt}\right)$$

$$+\frac{2}{7}(\sin\alpha + \cos\alpha\theta')\left(\mu + \frac{d\psi'}{dt}\right)\left(n + \mu\frac{c\sin\alpha - b}{a}\theta'\right)$$

$$= \frac{5}{7}\frac{g}{a}(\sin\alpha + \cos\alpha\theta').$$

This expression must be expanded and expressed in the form

$$\frac{d^2\theta'}{dt^2} + A\theta' = B,$$

in this case, since θ' contains only trigonometrical expressions, we must have $B = 0$. Hence putting $\theta' = 0$ in the above expression, we find

$$n = \frac{5}{2}\frac{g}{a\mu} + \frac{7\mu b}{a}\cot\alpha$$

as in steady motion.

After expanding the preceding equation we find

$$A = \mu^2\left(-\cos^2\alpha + \frac{2}{7}\sin^2\alpha\right) + \mu^2\frac{b}{c\sin\alpha}\left(2\cos^2\alpha + \frac{5}{7}\sin^2\alpha\right)$$

$$+\frac{25}{49}\frac{g^2\sin\alpha}{\mu^2 bc} - \frac{10}{7}\frac{g}{b}\sin\alpha\cos\alpha + \frac{10}{7}\frac{g}{c}\cos\alpha.$$

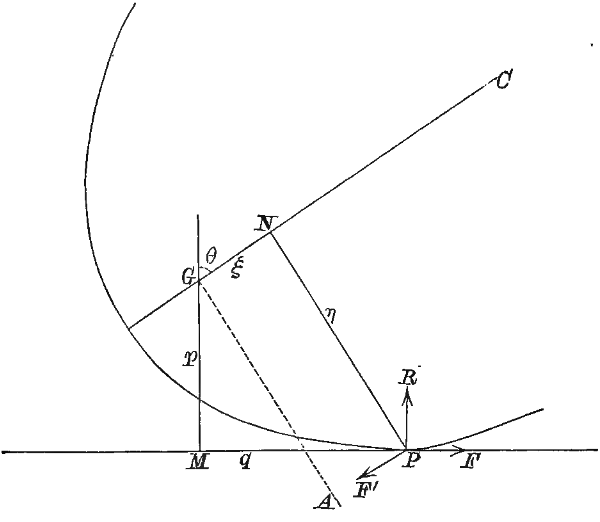
In order that the motion may be steady, it is sufficient and necessary that this value of A should be positive. And the time of oscillation is then $\frac{2\pi}{A}$.

It is to be observed that this investigation does not apply if a and therefore b be small, for some terms which have been rejected have b in their denominators, and may become important.

Motion of a Solid Body on a plane.

252. *A solid of revolution rolls on a perfectly rough horizontal plane under the action of gravity. To find the steady motion and the small oscillations.*

Let G be the centre of gravity of the body, GC the axis of figure, P the point of contact. Let GA be that principal axis which lies in the plane PGC and GB the axis at



right angles to GA, GC. Let GM be a perpendicular from G on the horizontal plane, and PN a perpendicular from P on GC. Let θ be the angle GC makes with the vertical, and ψ the angle MP makes with any fixed line in the horizontal plane. Let R be the normal reaction at P; F, F' the resolved parts of the frictions respectively in and perpendicular to the plane PGC.

Let us take moments about the moving axes GA, GB, GC according to Art. 121. Then $\theta_1 = 0$, $\theta_2 = 0$, and θ_3 is the angular velocity of the plane PGC about GC. We have

$$A\left(\frac{d\omega_1}{dt} - \theta_3\omega_2\right) - (A - C)\,\omega_2\omega_3 = -F' . GN \text{..............} (1),$$

$$A\left(\frac{d\omega_2}{dt} + \theta_3\omega_1\right) + (A - C)\,\omega_1\omega_3 = -F . GM - R . MP \text{....} (2),$$

$$C\frac{d\omega_3}{dt} = F' . PN \text{......................} (3).$$

The geometrical equations are

$$\frac{d\theta}{dt} = \omega_2 \text{..........................} (4),$$

$$\sin\theta\frac{d\psi}{dt} = -\omega_1 \text{.......................} (5).$$

$$-\theta_3 + \frac{d\psi}{dt}\cos\theta = \omega_3 \text{....................} (6).$$

Let u and v be the velocities of the centre of gravity respectively along and perpendicular to MP, both being parallel to the horizontal plane. The accelerations of the centre of gravity along these moving axes will be

$$\frac{du}{dt} - v\frac{d\psi}{dt} = F \text{.......................} (7),$$

$$\frac{dv}{dt} + u\frac{d\psi}{dt} = F' \text{.......................} (8).$$

And if z be the altitude of G above the horizontal plane, we have

$$\frac{d^2z}{dt^2} = -g + R \quad\text{..........................}\ (9).$$

Also since the point P is at rest, we have

$$u - GM\omega_2 = 0 \quad\text{......................}\ (10),$$

$$v + PN\omega_3 - GN\omega_1 = 0 \quad\text{......................}\ (11),$$

$$z = - GN \cos\theta + PN \sin\theta \quad\text{..................}\ (12).$$

When the motion is steady, we have the surface of revolution rolling on the plane so that its axis makes a constant angle with the vertical. In this state of motion, let $\theta = \alpha$, $\frac{d\psi}{dt} = \mu$, $\omega_3 = n$, $GM = p$, $MP = q$, $GN = \xi$, $NP = \eta$, and let ρ be the radius of curvature of the rolling body at P. Then the relations between these quantities may be found by substitution in the above equations.

Suppose it were required to find the conditions that the surface may roll with a given angular velocity n with its axis of figure making a given angle with the vertical. Here n and α are given, and p, q, ξ, η, ρ may be found from the equations to the surface. We have to find μ, ω_1, ω_2, u, v and the radius of the circle described by G in space. Then eliminating F, F', R, we get

$$\mu^2 \sin\alpha\,(A\cos\alpha - p\xi) - n\mu\,(C\sin\alpha + p\eta) - gq = 0,$$

$$\omega_1 = - \mu \sin\alpha,$$

$$\omega_2 = 0,$$

$$u = 0,$$

$$v = - n\eta - \xi\mu \sin\alpha.$$

Let r be the radius of the circle described by G as the surface rolls on the plane. Since G describes its circle with angular velocity μ, we have $r\mu = v$, and hence

$$r = -\frac{\eta n}{\mu} - \xi \sin\alpha.$$

For every value of n and α there are two values of μ, which however correspond to different initial conditions. In order that a steady motion may be possible, it is necessary that the roots of this quadratic should be real. This gives

$$(C\sin\alpha + p\eta)^2 n^2 + 4gq\sin\alpha\,(A\cos\alpha - p\xi)$$
$$= \text{a positive quantity.}$$

To find the small oscillation, we put $\theta = \alpha + \theta'$, $\dfrac{d\psi}{dt} = \mu + \dfrac{d\psi'}{dt}$, $\omega_3 = n + \omega'_3$. Then we have by geometry,

$$\begin{aligned} z = GM &= p + q\theta', \\ PM &= q + (\rho - p)\,\theta', \\ GN &= \xi + \rho\theta'\sin\alpha, \\ PN &= \eta + \rho\theta'\cos\alpha, \end{aligned}$$

and substituting in (5), (10), (11), (7), (8) respectively, we find

$$\omega_1 = -\mu\sin\alpha - \mu\cos\alpha\theta' - \sin\alpha\frac{d\psi'}{dt},$$

$$u = p\frac{d\theta'}{dt},$$

$$v = -\mu\sin\alpha\xi - n\eta - (\mu\cos\alpha\xi + \mu\rho\sin^2\alpha + n\rho\cos\alpha)\,\theta'$$
$$- \sin\alpha\xi\frac{d\psi'}{dt} - \eta\omega'_3,$$

$$F = p\frac{d^2\theta'}{dt^2} + \mu^2\sin\alpha\xi + n\mu\eta + 2\sin\alpha\mu\xi\frac{d\psi'}{dt} + \eta n\frac{d\psi'}{dt}$$
$$+ \mu(\mu\cos\alpha\xi + \mu\rho\sin^2\alpha + n\rho\cos\alpha)\,\theta' + \eta\mu\omega'_3,$$

$$F' = -(\mu\cos\alpha\xi - p\mu + \mu\rho\sin^2\alpha + n\rho\cos\alpha)\frac{d\theta'}{dt}$$
$$- \sin\alpha\xi\frac{d^2\psi'}{dt^2} - \eta\frac{d\omega'_3}{dt}.$$

Substituting these in equation (3) and integrating, we have

$$(C+\eta^2)\omega'_3 = (p\mu - \mu\xi\cos\alpha - \mu\rho\sin^2\alpha - n\rho\cos\alpha)\,\eta\theta' - \eta\sin\alpha\xi\frac{d\psi'}{dt} \quad \text{(A)},$$

the constant being omitted because n, α and μ are supposed to contain all the constant parts of ω_3, θ, and $\frac{d\psi}{dt}$.

Again substituting in (1) and integrating, we have

$$\{Cn - 2A\mu\cos\alpha + \xi(p\mu - \mu\cos\alpha\xi - \mu\sin^2\alpha\rho - n\rho\cos\alpha)\}\,\theta'$$
$$- (A + \xi^2)\sin\alpha\frac{d\psi'}{dt} = \xi\eta\omega'_3 \text{ .... (B).}$$

Also substituting in (2), we have

$$(A+p^2+q^2)\frac{d^2\theta'}{dt^2} + \theta'\{A\mu^2(\sin^2\alpha - \cos^2\alpha) + Cn\mu\cos\alpha + (\rho - p)g$$
$$+ \mu^2\sin\alpha\xi q + n\mu\eta q + \mu^2\cos\alpha\xi p + n\mu\rho p\cos\alpha + \mu^2\sin^2\alpha\rho p\}$$
$$+ \frac{d\psi'}{dt}\{-2A\mu\sin\alpha\cos\alpha + Cn\sin\alpha + 2\xi p\mu\sin\alpha + np\eta\}$$
$$+ \omega'_3\{C\mu\sin\alpha + \mu p\eta\}$$
$$+ \{-A\sin\alpha\cos\alpha\mu^2 + Cn\mu\sin\alpha + gq + \sin\alpha\mu^2 p\xi + n\mu p\eta\}$$
$$= 0 \text{ .... (C).}$$

The last term of this equation must vanish since θ', $\frac{d\psi'}{dt}$, ω'_3 only contain periodic terms. It is the equation thus formed which determines the steady motion and gives us the value of μ.

To solve these equations we may put

$$\left.\begin{aligned} \theta' &= L\sin(\lambda t + f) \\ \frac{d\psi'}{dt} &= M\sin(\lambda t + f) \\ \omega'_3 &= N\sin(\lambda t + f) \end{aligned}\right\}.$$

If we substitute these in (A), (B), (C) we shall get three equations to eliminate the ratios $L : M : N$. Before substitution it will be found convenient to simplify the equations first by multiplying (A) by ξ and (B) by η and subtracting the latter result from the former, and secondly by multiplying

(A) by $\frac{\mu p}{\eta}$ and adding the result to (C). We then obtain the following determinant,

$$\begin{vmatrix} \begin{array}{l} -(A+p^2+q^2)\lambda^2+(\rho-p)g \\ +\mu^2(p^2-A\cos 2\alpha-qr) \\ +n\mu C\cos\alpha \end{array} & \begin{array}{c} A\mu\sin\alpha\cos\alpha \\ +\frac{qg}{\mu} \end{array} & C\mu(\eta\sin\alpha-p) \\ Cn-2A\mu\cos\alpha & A\sin\alpha & C\xi \\ \begin{array}{l} (p-\xi\cos\alpha-\rho\sin^2\alpha)\mu \\ -\rho n\cos\alpha \end{array} & \xi\sin\alpha & -(C+\eta^2) \end{vmatrix} = 0.$$

253. EXAMPLE. *To find the least angular velocity which will make a hoop roll in a straight line.*

In this case r is infinite and therefore μ must be zero. It follows from the equation of steady motion that $q=0$ or the hoop must be upright. We have $p=a$, $q=0$, $\xi=0$, $\eta=a$, $\mu=0$, and $C=2A$. The determinant becomes

$$\lambda^2=\frac{2n^2(2A+a^2)-ag}{A+a^2},$$

so that the least angular velocity which will make λ a real quantity is given by $n^2=\frac{ag}{2(C+a^2)}$.

Let the hoop be an arc, we have $C=a^2$, and if V be the least velocity of the centre of gravity, this equation gives $V>\frac{1}{2}\sqrt{ag}$. Let the hoop be a disc, then $C=\frac{a^2}{2}$, and we have $V>\sqrt{\frac{ag}{3}}$.

254. *A surface of any form rolls on a perfectly rough horizontal plane under the action of gravity. To form the equations of motion.*

Let GA, GB, GC, the principal axes at the centre of gravity, be the axes of reference. Let $\phi(x, y, z) = 0$ be the equation to the bounding surface (x, y, z) the co-ordinates of the point of contact. Let (p, q, r) be the direction-cosines of the normal to the surface at the point x, y, z, then

$$\frac{p}{\frac{d\phi}{dx}} = \frac{q}{\frac{d\phi}{dy}} = \frac{r}{\frac{d\phi}{dz}}.$$

Let X, Y, Z be the resolved parts along the axes of the normal reaction and the two frictions at the point x, y, z, and let the mass of the body be unity.

By Euler's equations we have

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\,\omega_2\omega_3 &= yZ - zY \\ B\frac{d\omega_2}{dt} - (C-A)\,\omega_3\omega_1 &= zX - xZ \\ C\frac{d\omega_3}{dt} - (A-B)\,\omega_1\omega_2 &= xY - yX \end{aligned}\right\} \text{............(1).}$$

Also the equations of motion of the centre of gravity are by Art. 112,

$$\left.\begin{aligned} \frac{du}{dt} - v\omega_3 + w\omega_2 &= -gp + X \\ \frac{dv}{dt} - w\omega_1 + u\omega_3 &= -gq + Y \\ \frac{dw}{dt} - u\omega_2 + v\omega_1 &= -gr + Z \end{aligned}\right\} \text{.................(2).}$$

Also since the line (p, q, r) remains always vertical,

$$\left.\begin{aligned} \frac{dp}{dt} &= q\omega_3 - r\omega_2 \\ \frac{dq}{dt} &= r\omega_1 - p\omega_3 \\ \frac{dr}{dt} &= p\omega_2 - q\omega_1 \end{aligned}\right\} \text{........................(3).}$$

And since the point (x, y, z) is at rest,

$$\left.\begin{aligned} u - y\omega_3 + z\omega_2 = 0 \\ v - z\omega_1 + x\omega_3 = 0 \\ w - x\omega_2 + y\omega_1 = 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (4).$$

When the form of the surface is given we may be able in certain cases to solve these equations and to determine what states of steady motion are possible. Whatever the shape of the body may be we may suppose it to be set in rotation about the normal at the point of contact with an angular velocity n. If this angular velocity be not zero, the normal must be a principal axis at the point of contact, and yet it must pass through the centre of gravity. This cannot be unless the normal be a principal axis at the centre of gravity. If however $n = 0$, this condition is not necessary. There are therefore two cases to be considered.

Case 1. A body of any form is placed in equilibrium resting with the point C on a rough horizontal plane, with a principal axis at the centre of gravity vertical, and is then set in rotation with an angular velocity n about GC. A small disturbance being given to the body, it is required to find the motion.

Case 2. A body of any form is placed in equilibrium on a rough horizontal plane with the centre of gravity over the point of contact. A small disturbance being given to the body, to find the motion.

255. *Case* 1. Supposing the body not to depart far from its initial position, we have $p, q, u, v, w, \omega_1, \omega_2$ all small quantities and $r = 1$ nearly. Hence by (2), when we neglect the squares of small quantities, we see that X, Y are also small and $Z = g$ nearly. It follows by (1) that ω_3 is constant and $\therefore = n$. Also x and y are small and $z = -h$ nearly, where h is the altitude of the centre of gravity above the horizontal plane before the motion was disturbed. The equation to the surface may, by Taylor's theorem, be written in the form

$$z = -h + \frac{1}{2}\left(\frac{x^2}{a} + \frac{2xy}{b} + \frac{y^2}{c}\right),$$

where (a, b, c) are some constants depending on the curvatures of the principal sections of the body at the point C.

The squares of all small quantities being neglected, the preceding equations become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\,n\omega_2 &= gy + hY \\ B\frac{d\omega_2}{dt} - (C - A)\,n\omega_1 &= -hX - gx \end{aligned}\right\},$$

$$\left.\begin{aligned} \frac{du}{dt} - nv &= -gp + X \\ \frac{dv}{dt} + nu &= -gq + Y \end{aligned}\right\},$$

$$\left.\begin{aligned} \frac{dp}{dt} &= nq - \omega_2 \\ \frac{dq}{dt} &= \omega_1 - np \end{aligned}\right\},$$

$$\left.\begin{aligned} u - ny - h\omega_2 &= 0 \\ v + h\omega_1 + nx &= 0 \end{aligned}\right\},$$

$$\frac{p}{\dfrac{x}{a} + \dfrac{y}{b}} = \frac{q}{\dfrac{x}{b} + \dfrac{y}{c}} = -1.$$

Eliminating $X, Y, u, v, \omega_1, \omega_2$ from these equations, we get

$$(A + h^2)\frac{d^2q}{dt^2} + (A + B + 2h^2 - C)\,n\frac{dp}{dt} - \{(B - C)\,n^2 + hg + h^2n^2\}\,q$$
$$= (g + hn^2)\,y - hn\frac{dx}{dt}$$

$$-(B + h^2)\frac{d^2p}{dt^2} + (A + B + 2h^2 - C)\,n\frac{dq}{dt} + \{(A - C)\,n^2 + hg + h^2n^2\}\,p$$
$$= -(g + hn^2)\,x - hn\frac{dy}{dt}.$$

It will be found convenient to express x, y in terms of p, q. The right-hand sides of each of these equations will then take the form

$$Lp + Mq + L'\frac{dp}{dt} + M'\frac{dq}{dt}.$$

To solve these equations, we must then assume p, q to be of the form

$$\left.\begin{aligned} p &= P_0 \cos \lambda t + P_1 \sin \lambda t \\ q &= Q_0 \cos \lambda t + Q_1 \sin \lambda t \end{aligned}\right\}.$$

If the tangents to the lines of curvature of the moving body at C be parallel to the principal axes at the centre of gravity, these equations admit of considerable simplification. In that case the equation to the surface may be written in the form

$$z = -h + \frac{1}{2}\left(\frac{x^2}{a} + \frac{y^2}{c}\right),$$

where a and c are the radii of curvature of the lines of curvature. The right-hand sides of the equations then become respectively

$$-(g + hn^2)\,cq + hna\frac{dp}{dt},$$

$$(g + hn^2)\,ap + hnc\frac{dq}{dt}.$$

To satisfy the equations, it will be sufficient to put

$$\begin{aligned} p &= F\cos(\lambda t + f), \\ q &= G\sin(\lambda t + f). \end{aligned}$$

This simplification is possible, because we can see beforehand that $\dfrac{P_0}{P_1} = \dfrac{Q_0}{Q_1}$.

Substituting and eliminating the ratio $\dfrac{F}{G}$, we get the following quadratic to determine λ^2.

$$\begin{aligned}\left[(A + h^2)\lambda^2 + \{B - C + h(h - c)\}n^2 + g(h - c)\right]&\left[(B + h^2)\lambda^2\right. \\ &\left.+ \{A - C + h(h - a)\}n^2 + g(h - a)\right] \\ = \lambda^2 n^2 \{A + B + 2h^2 - C - ha\}&\{A + B + 2h^2 - C - hc\}.\end{aligned}$$

If λ_1, λ_2 be the roots of this equation, the motion is represented by the equations

$$\left.\begin{aligned} p &= F_1 \cos(\lambda_1 t + f_1) + F_2 \cos(\lambda_2 t + f_2) \\ q &= G_1 \sin(\lambda_1 t + f_1) + G_2 \sin(\lambda_2 t + f_2) \end{aligned}\right\},$$

where $\frac{G_1}{F_1}, \frac{G_2}{F_2}$ are known functions of λ_1, λ_2 respectively, and F_1, F_2, f_1, f_2 are constants to be determined by the initial values of $p, q, \frac{dp}{dt}, \frac{dq}{dt}$.

In order that the motion may be stable, it is necessary that the roots of this quadratic should be real and positive. These conditions may be easily expressed.

256. *Case* 2. Supposing the disturbance to be small, we have $\omega_1, \omega_2, \omega_3, u, v, w$ all small quantities. Hence when we neglect the squares of small quantities the equations (1) and (2) become respectively.

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} &= yZ - zY \\ B\frac{d\omega_2}{dt} &= zX - xZ \\ C\frac{d\omega_3}{dt} &= xY - yX \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$\left.\begin{aligned} \frac{du}{dt} &= -gp + X \\ \frac{dv}{dt} &= -gq + Y \\ \frac{dw}{dt} &= -gr + Z \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Let x_0, y_0, z_0 be the co-ordinates of the point of contact in the position of equilibrium, and let

$$x = x_0 + x', \quad y = y_0 + y', \quad z = z_0 + z'.$$

Then in the small terms of equation (4) we may write x_0, y_0, z_0 for x, y, z. Hence differentiating these and eliminating X, Y, Z, u, v, w by help of equations (i) and (ii), we get

$$(A + y_0^2 + z_0^2)\frac{d\omega_1}{dt} - x_0y_0\frac{d\omega_2}{dt} - x_0z_0\frac{d\omega_3}{dt} = g\,(yr - zq) \ldots \text{(iii)},$$

and two similar equations.

Let p_0, q_0, r_0 be the values of p, q, r in the position of equilibrium. Then $\frac{x_0}{p_0} = \frac{y_0}{q_0} = \frac{z_0}{r_0} = \rho$, where ρ is the radius vector from G to the point of contact. Now in the small terms of equations (3) we may write p_0, q_0, r_0 for p, q, r. Hence equations (iii) become by substitution

$$A\frac{d\omega_1}{dt} = y_0\rho\frac{d^2r}{dt^2} - z_0\rho\frac{d^2q}{dt^2} + g\,(yr - zq) \ldots\ldots\ldots \text{(iv)},$$

and two similar equations.

At the time t let $p = p_0 + p'$, $q = q_0 + q'$, and $r = r_0 + r'$.

Then since

$$(p_0 + p')^2 + (q_0 + q')^2 + (r_0 + r')^2 = 1,$$

we have

$$p_0p' + q_0q' + r_0r' = 0.$$

The form of the surface being known we can find p', q', r' in terms of x', y', z', and thus express $yr - zq$, $zp - xr$, $xq - yp$ in the form

$$g\,(yr - zq) = Lp' + Mq'.$$

The equations (iv) now become

$$A\frac{d\omega_1}{dt} = y_0\rho\frac{d^2r'}{dt^2} - z_0\rho\frac{d^2q'}{dt^2} + Lp' + Mq' \ldots\ldots\ldots \text{(v)},$$

and two similar equations.

Differentiating equations (3), and substituting for $\frac{d\omega_1}{dt}$, $\frac{d\omega_2}{dt}$, $\frac{d\omega_3}{dt}$, r' and $\frac{d^2r'}{dt^2}$, we get equations of the form

$$\left.\begin{aligned} F\frac{d^2p'}{dt^2} + G\frac{d^2q'}{dt^2} &= Hp' + Kq' \\ F'\frac{d^2p'}{dt^2} + G'\frac{d^2q'}{dt^2} &= H'p' + K'q' \end{aligned}\right\}.$$

To solve these we put

$$p' = P\cos(\lambda t + f),$$
$$q' = Q\cos(\lambda t + f),$$

substituting and eliminating the ratios $\frac{P}{Q}$, we have the quadratic

$$\begin{vmatrix} F\lambda^2 + H, & G\lambda^2 + k \\ F'\lambda^2 + H', & G'\lambda^2 + k' \end{vmatrix} = 0 \text{ .... (vi)},$$

to determine λ^2.

Thus by virtue of the relation existing between p', q', r', each of these may be represented by an expression of the form

$$P_1\cos(\lambda_1 t + f_1) + P_2\cos(\lambda_2 t + f_2).$$

Substituting these values in equations (v) we see that ω_1, ω_2, ω_3 can each be represented by an expression

$$\Omega_1 + E_1\cos(\lambda_1 t + f_1) + E_2\cos(\lambda_2 t + f_2),$$

where E_1, E_2 are known functions of P_1, P_2 ... and λ_1, λ_2, but Ω_1, Ω_2, Ω_3 are small arbitrary quantities. By substituting in equations (3) and equating the coefficients of $\cos(\lambda_1 t + f_1)$ and $\cos(\lambda_2 t + f_2)$, we may find the values of E_1 and E_2 without difficulty. And we also see that we must have

$$\frac{\Omega_1}{p_0} = \frac{\Omega_2}{q_0} = \frac{\Omega_3}{r_0},$$

so that, of the three Ω_1, Ω_2, Ω_3, only one is really arbitrary. We have therefore but five arbitrary constants, viz. P_1, P_2, f_1, f_2, and Ω_1. These are determined by the initial values of ω_1, ω_2, ω_3, p' and q'.

To find the motion of the principal axes round the vertical, let ϕ be the angle the plane containing GC and the vertical makes with the plane of AC. Then, by drawing a figure for the standard case in which p, q, r are all positive, it will be seen that if μ be the rate at which GC goes round the vertical,

$$\mu\sqrt{1-r^2} = \omega_1 \cos\phi + \omega_2 \sin\phi$$

$$= \frac{p_0\omega_1 + q_0\omega_2}{\sqrt{1-r_0^2}}.$$

Substituting for ω_1, ω_2, this takes the form

$$\mu = n_3 + N_1 \cos(\lambda_1 t + f_1) + N_2 \cos(\lambda_2 t + f_2),$$

where n_3, N_1, N_2 are all known constants.

In order that the equilibrium may be stable it is necessary that the roots of the quadratic (vi) should both be real and positive. These conditions may easily be expressed.

These conditions being supposed satisfied, the expressions for p', q', r' will only contain periodical terms, and thus the inclinations of the principal axes to the vertical will not be sensibly altered. But the expressions for ω_1, ω_2, ω_3 *may* each contain a non-periodical term, and if so the rate at which the principal axes will go round the vertical will also contain non-periodical terms. The body therefore may gradually turn with a slow motion round the normal at the point of contact. The expressions for u, v, w will contain only periodic terms, so that the body will have no motion of translation in space.

Motion of a Rod.

257. When the body whose motion is to be determined is a rod, it is often more convenient to recur to the original equations of motion supplied by D'Alembert's Principle. The equations of Lagrange may also be used with advantage. These methods will be illustrated by the following problem.

A uniform heavy rod, suspended from a fixed point O *by a string, makes small oscillations about the vertical. Determine the motion.*

Let O be taken as origin, and let the axis of z be measured vertically downwards; let $2a$ be the length of the rod,

b the length of the string. Let (l, m, n) (p, q, r) be the direction-cosines of the string and rod. Then l, m, p, q are small quantities whose squares are to be neglected and we may put n and r each equal to unity. Let u be the distance of any element du of the rod from that extremity A of the rod to which the string is attached. Let (x, y, z) be the co-ordinates of the element du, then we have

$$\left.\begin{aligned} x &= bl + up \\ y &= bm + uq \\ z &= b \quad + u \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (1).$$

Let M be the mass of the rod, MT the tension of the string, the equations of motion of the centre of gravity will be

$$\left.\begin{aligned} b\frac{d^2l}{dt^2} + a\frac{d^2p}{dt^2} &= -Tl \\ b\frac{d^2m}{dt^2} + a\frac{d^2q}{dt^2} &= -Tm \\ 0 &= g - T \end{aligned}\right\} \dots\dots\dots\dots (2).$$

By D'Alembert's Principle the equation of moments round x will be

$$\Sigma du\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = \Sigma du\,(yZ - zY)$$
$$= \Sigma du\,(yg).$$

By equations (1) this reduces to

$$\int_0^{2a} du\left\{-(b+u)\left(b\frac{d^2m}{dt^2} + u\frac{d^2q}{dt^2}\right)\right\} = 2ag\,(bm + aq).$$

Integrating, we get

$$-2ab\left(b\frac{d^2m}{dt^2} + a\frac{d^2q}{dt^2}\right) - 2ba^2\frac{d^2m}{dt^2} - \frac{8a^3}{3}\frac{d^2q}{dt^2} = 2ag\,(bm + aq),$$

which by equations (2) reduces to

$$b\frac{d^2m}{dt^2} + \frac{4}{3}a\frac{d^2q}{dt^2} = -gq.$$

Therefore the four equations of motion are

$$\left.\begin{aligned} b\frac{d^2l}{dt^2}+a\frac{d^2p}{dt^2} &= -gl \\ b\frac{d^2l}{dt^2}+\frac{4}{3}a\frac{d^2p}{dt^2} &= -gp \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and two similar equations for m, q. These equations do not contain m or q, and on the other hand the equations to find m and q do not contain l or p. This shows that the oscillations in the plane xz are not affected by those in the perpendicular plane yz. See Art. 198.

To solve these equations, put

$$l = F\sin(\lambda t+\alpha), \quad p = G\sin(\lambda t+\alpha),$$

we get

$$\left.\begin{aligned} b\lambda^2 F + a\lambda^2 G &= gF \\ b\lambda^2 F + \frac{4}{3}a\lambda^2 G &= gG \end{aligned}\right\};$$

$$\therefore\ \lambda^4 - \frac{4a+3b}{ab}g\lambda^2 + \frac{3g^2}{ab} = 0,$$

and the values of λ may be found from this equation.

In order to make a comparison of different methods, let us deduce the motion from Lagrange's equations. In this case we must determine the semi vis viva T true to the *squares* of the small quantities p, q, l, m, we cannot therefore put $r=1$, $n=1$. Since $p^2+q^2+r^2=1$, $l^2+m^2+n^2=1$, we have

$$\left.\begin{aligned} r &= 1-\frac{p^2+q^2}{2} \\ n &= 1-\frac{l^2+m^2}{2} \end{aligned}\right\},$$

we must therefore replace the third of equations (1) by

$$\begin{aligned} z &= bn+ur \\ &= b+u-b\frac{l^2+m^2}{2}-u\frac{p^2+q^2}{2}. \end{aligned}$$

If accents denote differential coefficients with regard to t, as in Lagrange's equations we have

$$\Sigma m x'^2 = \Sigma m\,(b^2 l'^2 + 2bl'p'u + p'^2 u^2)$$

$$= M\left(b^2 l'^2 + 2bl'p'a + \frac{4a^2}{3} p'^2\right).$$

The value of $\Sigma m y'^2$ may be found in a similar manner. The value of $\Sigma m z'^2$ is of the fourth order and may be neglected. Hence we have

$$2T = b^2 (l'^2 + m'^2) + 2ab\,(l'p' + m'q') + \frac{4a^2}{3} (p'^2 + q'^2).$$

Also we have

$$U = -g\left(b\,\frac{l^2 + m^2}{2} + a\,\frac{p^2 + q^2}{2}\right) + \text{constant}.$$

The equation

$$\frac{d}{dt}\frac{dT}{dl'} - \frac{dT}{dl} = \frac{dU}{dl}$$

becomes

$$bl'' + ap'' = -gl;$$

similarly we get

$$bl' + \frac{4a}{3} p'' = -gp.$$

These are the same equations which we deduced from D'Alembert's Principle and the solution may be continued as before.

EXAMPLES.

1. A body turning about a fixed point is acted on by forces which tend to *produce* rotation about an axis at right angles to the instantaneous axis, show that the angular velocity cannot be uniform unless the momental ellipsoid at the fixed point is a spheroid.

The axis about which the forces tend to produce rotation is that axis about which it would begin to turn if the body were placed at rest.

2. If forces act on a homogeneous spheroid tending always to produce rotation about an axis (α) in the plane of the equator, the instantaneous axis will describe a circular cone in the body about its polar axis; but the angular velocity about the instantaneous axis will not be uniform unless the axis α be always at right angles to the instantaneous axis.

3. A top two of whose principal moments at O are equal is set in rotation about its axis of figure viz. OC with an angular velocity n, the point O being fixed. If OC be *horizontal*, and if the proper initial angular velocity be communicated to the top about the vertical through O, prove that the top will not fall down, but that the axis of figure will revolve round the vertical, in steady motion, with an angular velocity $\mu = \frac{gh}{Cn}$, where h is the distance of the centre of gravity of the top from O, and C is the moment of inertia about the axis of figure. Show also that if the top be initially placed with OC nearly horizontal and if a very great angular velocity be communicated to it about OC without any initial angular velocity about OA or OB, then OC will revolve round the vertical remaining very nearly in a horizontal plane with an angular velocity μ given by the same formula as before, and the time of the vertical oscillations of OC about its mean position will be

$$2\pi \frac{A}{\sqrt{C^2n^2 + Agh}}.$$

4. A uniform rod, moveable about one extremity, moves in such a manner as to make always nearly the same angle α with the vertical; show that the time of a small oscillation is

$$2\pi \sqrt{\frac{2a}{3g} \cdot \frac{\cos\alpha}{1 + 3\cos^2\alpha}},$$

a being the length of the rod.

5. A homogeneous sphere is placed upon an inclined plane sufficiently rough to prevent sliding and a velocity in

any direction is communicated to it. Show that the path of its centre will be a parabola, and if V be the initial horizontal velocity of the centre of gravity, α the inclination of the plane to the horizon, the latus rectum will be $\frac{14}{5}\frac{V^2}{g \sin \alpha}$.

These results may be deduced from Art. 247. We may also proceed thus. Taking the plane as the plane of xy and the axis of x down the plane, the equations of Art. 107, become

$$k^2\frac{d\omega_1}{dt} = F'a,\quad k^2\frac{d\omega_2}{dt} = -Fa,\quad k^2\frac{d\omega_3}{dt} = 0,\quad \frac{dv_x}{dt} = g\sin\alpha + F,\quad \frac{dv_y}{dt} = F';$$

also the geometrical equations are $v_x - a\omega_2 = 0$, $v_y + a\omega_1 = 0$, where F, F' are the frictions parallel to the axes of x and y respectively, a is the radius of the sphere and the mass has been taken unity. Differentiating the geometrical relations and substituting from the dynamical equations we can find F and F'. Thence v_x and v_y may be found.

6. If a rough plane inclined at an angle α to the horizon be made to revolve with uniform angular velocity n about a normal Oz and a sphere be placed at rest upon it, show that the path in space of the centre will be a prolate, a common, or a curtate cycloid according as the point at which the sphere is initially placed is without, upon, or within the circle whose equation is

$$x^2 + y^2 = \frac{35 g \sin \alpha}{2n^2} x,$$

the axis Oy being horizontal.

Take the same co-ordinate axes as in the last question. The equations may be treated in a similar manner. When the sphere is placed at rest on the moving plane, a velocity is suddenly given to it by the impulsive frictions.

7. A circular disc capable of motion about a vertical axis through its centre perpendicular to its plane is set in motion with angular velocity Ω. A rough uniform sphere is gently placed on any point of the disc, not the centre, prove that the sphere will describe a circle on the disc, and that the disc will revolve with angular velocity $\frac{7M}{7M+4m}\Omega$ where M and m are the masses of the disc and sphere.

8. A sphere is pressed between two perfectly rough parallel boards which are made to revolve with the uniform

angular velocities Ω and Ω' about fixed axes perpendicular to their planes. Prove that the centre of the sphere describes a circle about an axis which is in the same plane as the axes of revolution of the boards and whose distances from these axes are inversely proportional to the angular velocities about them.

Show that when the boards revolve about the same axis, their points of contact will trace on the sphere small circles, the tangents of whose angular radii will be $\frac{c}{a} \cdot \frac{\Omega \sim \Omega'}{\Omega + \Omega'}$, a being the radius of the sphere and c that of the circle described by its centre.

9. Prove that a billiard ball, being considered as a sphere partly rolling and partly sliding on a horizontal plane, describes a parabola, except in the particular case when it moves in a right line.

10. A perfectly rough circular cylinder is fixed with its axis horizontal. A sphere being placed on it in a position of unstable equilibrium is so projected that the centre begins to move with a velocity V parallel to the axis of the cylinder. It is then slightly disturbed in a direction perpendicular to the axis. If θ be the angle the radius through the point of contact makes with the vertical, prove that the velocity of the centre parallel to the axis at any time t is $V \cos \sqrt{\frac{2}{7}}\, \theta$ and that the sphere will leave the cylinder when $\cos \theta = \frac{10}{17}$.

11. A uniform sphere is placed in contact with the exterior surface of a perfectly rough cone. Its centre is acted on by a force the direction of which always meets the axis of the cone at right angles and the intensity of which varies inversely as the cube of the distance from that axis. Prove that if the sphere be properly started the path described by its centre will meet every generating line of the cone on which it lies in the same angle.

See the solutions of *Cambridge Problems* for 1860, page 92.

12. A perfectly rough sphere is placed on another perfectly rough fixed sphere near the highest point. The upper sphere has an angular rotation ω about its diameter through the point of contact; prove that its equilibrium will be stable if $\omega^2 > \frac{35g\,(a+b)}{a^2}$ where b is the radius of the fixed sphere, and a the radius of the moving sphere.

13. Every particle of a sphere of radius a, which is placed on a perfectly rough sphere of radius c, is attracted to a centre of force on the surface of the fixed sphere with a force varying inversely as the square of the distance; if it be placed at the extremity of the diameter through the centre of force and be set rotating about that diameter and then slightly displaced, determine its motion; and show that when it leaves the fixed sphere the distance of its centre from the centre of force is a root of the equation

$$20x^3 - 13\,(2c+a)\,x^2 + 7a\,(2c+a)^2 = 0.$$

14. A perfectly rough plane revolves uniformly about a vertical axis in its own plane with an angular velocity n, a sphere being placed in contact with the plane rolls on it under the action of gravity find the motion.

Take the axis of revolution as axis of z, and let the axis of x be fixed in the plane. Let a be the radius, m the mass of the sphere; F, F' the frictions resolved parallel to the axes of x and z and R the normal reaction. The equations of motion are therefore by Art. 94 $\frac{d^2x}{dt^2} - n^2x = \frac{F}{m}$, $-an^2 + 2n\frac{dx}{dt} = \frac{R}{m}$ and $\frac{d^2z}{dt^2} = -g + \frac{F'}{m}$. The equations of rotation by Art. 122 are $\frac{d\omega_x}{dt} - n\omega_y = -\frac{F'a}{k^2}$, $\frac{d\omega_y}{dt} + n\omega_x = 0$, $\frac{d\omega_z}{dt} = \frac{Fa}{k^2}$. Since the point of contact has the same motion as the plane the geometrical equations by Art. 111 are $\frac{dx}{dt} - an + a\omega_z = 0$, $\frac{dz}{dt} - a\omega_x = 0$. Solving these equations we find that the sphere will not fall down. If the sphere start from *relative* rest at a point in the axis of x, we have $z = -\frac{g}{n^2}\tan^2 i\,\{1 - \cos(nt\cos i)\}$ where $\sin i = \sqrt{\frac{5}{7}}$. The sphere will therefore never descend more than $\frac{5g}{n^2}$ below its original position.

15. A sphere of radius a is suspended from a fixed point O by a string of length b and is made to rotate about a

vertical axis with an angular velocity n. Prove that, if the string make small oscillations about its mean position, the motion of the centre of gravity will be represented by a series of terms of the form $L\cos(\lambda t + M)$ where the several values of λ are the roots of the equation

$$(b\lambda^2 - g)\left(\lambda^2 - n\lambda - \frac{5g}{2a}\right) = \frac{5}{2}g\lambda^2.$$

Let C be the point of the sphere to which the string is attached, G the centre of gravity. Take as axes of reference GC and two axes GA, GB fixed in the body. Let (p, q, r) be the direction-cosines of the vertical and (p', q', r') those of CO. Let (u, v, w) be the velocities of G parallel to the axes. Then $u, v, \omega_1, \omega_2, p, q, p', q'$ are all small quantities. Also $w=0$, $\omega_3=n$, $r=1$, $r'=1$. By Euler's equations we have $k^2\frac{d\omega_1}{dt} = -gaq'$, $k^2\frac{d\omega_2}{dt} = gap'$. By Art. 112 we have $\frac{du}{dt} - nv = g(p'-p)$, $\frac{dv}{dt} + nu = g(q'-q)$. By Art. 114 we can express the fact that O is fixed in space viz. $u + b\frac{dp'}{dt} - bnq' + (a+b)\omega_2 = 0$, $v + b\frac{dq'}{dt} - (a+b)\omega_1 + bnp' = 0$. By Art. 113 we can express the fact that gravity acts vertically, this gives $\frac{dp}{dt} - nq + \omega_2 = 0$, $\frac{dq}{dt} - \omega_1 + np = 0$. All these equations are linear and may be solved in the usual manner.

16. An infinitely thin circular disc moves on a perfectly rough horizontal plane in such a manner as to preserve a constant inclination α to the vertical. Find the condition that the motion may be steady and the time of a small oscillation.

Let the radius of the disc be a, and the radius of gyration about a diameter k. Let ω_3 be the angular velocity about the axis, μ the angular velocity of the centre of gravity about the centre of the circle described by it, r the radius of this circle, then in steady motion

$$(2k^2 + a^2)\,\omega_3 = k^2\mu\cos\alpha - \frac{ga}{\mu}\cot\alpha,$$

$$(2k^2 + a^2)\,r = -k^2 a\cos\alpha + \frac{ga^2}{\mu^2}\cot\alpha.$$

If T be the time of a small oscillation

$$\left(\frac{2\pi}{T}\right)^2 (k^2 + a^2) = \mu^2\{k^2(1 + 2\cos^2\alpha) + a^2\sin^2\alpha\}$$
$$- 2n\mu\cos\alpha\,(3k^2 + a^2) + 2n^2(2k^2 + a^2) - ga\sin\alpha.$$

17. An ellipsoid is placed with one of its vertices in contact with a smooth horizontal plane. What angular velocity of rotation must it have about the vertical axis in order that the equilibrium may be stable?

Result. Let a, b, c be the semi-axes, c the vertical axis, then the angular velocity must be greater than

$$\sqrt{\frac{5g}{c}\cdot\frac{\sqrt{c^4-a^4}+\sqrt{c^4-b^4}}{a^2+b^2}}.$$

18. A surface of revolution rolls on another perfectly rough surface of revolution with its axis vertical. The centre of gravity of the rolling surface lies in its axis. Find the cases of steady motion in which it is possible for the axes of both the surfaces to lie in a vertical plane throughout the motion.

Let θ be the inclination of the axes of the two surfaces, P the point of contact, GM a perpendicular on the tangent plane at P, PN a perpendicular on the axis GC of the rolling body; F the friction, R the reaction at P; n the angular velocity of the rolling body about its axis GC, μ the angular rate at which G describes its circular path in space, r the radius of this circle. Then in steady motion

$$M\mu\sin\theta\,(Cn-A\mu\cos\theta)=-F\,.\,GM-R\,.\,MP$$

$$R=-Mr\mu^2\sin\alpha+Mg\cos\alpha,$$

$$F=-Mr\mu^2\cos\alpha-Mg\sin\alpha,$$

$$n\,.\,PN+\mu\sin\theta\,.\,GN=-r\mu,$$

where M is the mass of the body.

19. A hoop $AGBF$ revolves about AB its diameter as a fixed vertical axis. GF is a horizontal diameter of the same circle which is without mass and which is rigidly connected to the circle; DC is a smaller concentric hoop which can turn freely about GF as diameter. If Ω, Ω', ω, ω', be the greatest and least angular velocities about AB, GF respectively, prove that

$$\Omega\,.\,\Omega'=\omega^2-\omega'^2.$$

20. OA, OB, OC are the principal axes of a rigid body which is in motion about a fixed point O. The axis OC has a constant inclination α to a line OZ fixed in space, and revolves with uniform angular velocity Ω round it, and the axis OA always lies in the plane ZOC. Prove that the constraining couple has its axis coincident with OB, and that its moment is $-(A-C)\,\Omega^2\sin\alpha\cos\alpha$.

21. If a rigid body be moving in any manner in three dimensions, prove that we can take moments about the instantaneous axis as if it were an axis fixed in space and in the body (Art. 25), provided the moment of inertia of the body about the instantaneous axis is constant throughout the motion.

We have by Art. 107 the equation

$$A\frac{d\omega_1}{dt}-(B-C)\,\omega_2\omega_3=L,$$

and two similar equations. Let (l, m, n) be the direction-cosines of the instantaneous axis, I and Ω the moment of inertia and angular velocity about it. Then $\omega_1=l\Omega$, $\omega_2=m\Omega$, $\omega_3=n\Omega$. Substituting in the above equations we get

$$Al^2\frac{d\Omega}{dt}-(B-C)\,lmn\Omega^2=lL-Al\frac{dl}{dt}\,\Omega,$$

and two other equations. Adding these three we get

$$I\frac{d\Omega}{dt}=P-\Omega\left(Al\frac{dl}{dt}+Bm\frac{dm}{dt}+Cn\frac{dn}{dt}\right),$$

where P is the moment of all the forces about the instantaneous axis. Hence the equation

$$I\frac{d\Omega}{dt}=P,$$

will be true whenever $Al^2+Bm^2+Cn^2$ is constant throughout the motion.

22. A right cone is placed with its slant side on a perfectly rough plane and rolls on it under the action of gravity. Find the motion.

Here as the cone rolls the instantaneous axis makes a constant angle with the axis, the moment of inertia about it is therefore constant. We may therefore take moments about the generating line in contact with the plane as if it were fixed. Let the fixed vertex O of the cone be the origin, and let ϕ be the angle the side of the cone in contact with the plane makes with the direction in which gravity acts when resolved along the plane. Let 2α be the angle,

h the length of the axis of the cone, β the inclination of the plane to the horizon. Then

$$I\frac{d\Omega}{dt} = -Mg\sin\beta\,\frac{3}{4}h\sin\alpha\sin\phi.$$

If C be the centre of the base of the cone, the velocity of C due to an angular velocity Ω about the generating line is $h\sin\alpha\,\Omega$. If CN be a perpendicular from C on the instantaneous axis, the velocity of C in space is also equal to the velocity of N in space, because C is just over N. But since $ON = h\cos\alpha$, the velocity of N is $h\cos\alpha\frac{d\phi}{dt}$, $\therefore\ \Omega\sin\alpha = \frac{d\phi}{dt}\cos\alpha$. The equation of motion is therefore

$$\frac{d^2\phi}{dt^2} = -\frac{3}{4}\,\frac{Mgh\sin^2\alpha\sin\beta}{I\cos\alpha}\sin\phi.$$

This equation can be once integrated and the angular velocity found. If the cone *just* make complete revolutions, the angular velocity at the lowest point will be given by

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{10g\sin\beta}{h}\cdot\frac{\cos\alpha}{\sin^2\alpha + 6\cos^2\alpha}.$$

In the same way we may find the motion of a *right* cone rolling under the action of any forces on another perfectly rough cone; or the vertex being fixed, rolling on any rough curve. These motions may also be deduced from the principle of Vis Viva. Thus the vis viva due to an angular velocity Ω about the instantaneous axis is $I\Omega^2$. Also the force function is easily found to be $Mg\sin\beta\,\frac{3}{4}h(1-\cos\phi)\cos\alpha$. Hence

$$I\Omega^2 = -Mg\sin\beta\,\frac{3}{2}h\cos\phi\cos\alpha + \text{constant}.$$

Substituting for Ω its value in terms of ϕ we may proceed as before.

23. A hollow cone, the internal surface of which is perfectly rough, is fixed in a position in which its axis is inclined at an angle (α) to the vertical, and a solid cone of smaller vertical angle is placed inside, its vertex coinciding with the vertex of the fixed cone, and allowed to oscillate: show that the length of the simple isochronous pendulum is

$$\frac{4k^2}{3h}\,\frac{\sin(\beta-\gamma)}{\sin\alpha\sin^2\gamma},$$

2β and 2γ being the vertical angles, h the height of the moving cone, and k its radius of gyration about a generating line.

CHAPTER XI.

PRECESSION AND NUTATION,
&c. &c.

On the Potential.

258. *To find the potential of a body of any form at any external distant point.*

Let the centre of gravity G of the body be taken as the origin of co-ordinates and let the axis of x pass through S the external point. Let the distance $GS=\rho$. Let (x, y, z) be the co-ordinates of any element dm of the body situated at any point P and let $GP=r$. Then the potential of the body is

$$V=\Sigma \frac{dm}{PS}.$$

Now $$PS^2=\rho^2+r^2-2\rho x;$$

$$\therefore\ V=\Sigma \frac{dm}{\rho}\left\{1-\frac{2\rho x-r^2}{\rho^2}\right\}^{-\frac{1}{2}}$$

$$=\Sigma \frac{dm}{\rho}\left\{1+\frac{1}{2}\,\frac{2\rho x-r^2}{\rho^2}+\frac{3}{8}\left(\frac{2\rho x-r^2}{\rho^2}\right)^2\right.$$

$$\left.+\frac{5}{16}\left(\frac{2\rho x-r^2}{\rho^2}\right)^3+\frac{35}{128}\left(\frac{2\rho x-r^2}{\rho^2}\right)^4+\dots\right\},$$

arranging these terms in descending powers of ρ, we get

$$V = \Sigma \frac{dm}{\rho} \left\{ 1 + \frac{x}{\rho} + \frac{3x^2 - r^2}{2\rho^2} + \frac{5x^3 - 3xr^2}{2\rho^3} + \frac{35x^4 - 30x^2r^2 + 3r^4}{8\rho^4} + \dots \right\}.$$

Let M be the mass of the body, then $\Sigma dm = M$.

Also since the origin is at the centre of gravity, we have $\Sigma x dm = 0$.

Let A, B, C be the principal moments of inertia at the centre of gravity, I the moment of inertia about the axis of x, which in our case is the line joining the centre of gravity of the body to the attracted point. Then

$$\Sigma dm\, r^2 = \frac{1}{2}(A + B + C),$$

$$\Sigma dm\, x^2 = \Sigma dm\, (r^2 - y^2 - z^2),$$

$$= \frac{1}{2}(A + B + C) - I.$$

Let l be any linear dimension of the body, then if ρ be so great compared with l that we may neglect the fraction $\left(\frac{l}{\rho}\right)^3$ of the potential, we have

$$V = \frac{M}{\rho} + \frac{A + B + C - 3I}{2\rho^3}.$$

If we wish to make a nearer approximation to the value of V, we must take account of the next terms, viz.

$$\frac{5\Sigma mx^3 - 3\Sigma mxr^2}{2\rho^4}.$$

Let (ξ, η, ζ) be the co-ordinates of m referred to any fixed rectangular axes having the origin at G, and let (α, β, γ) be the angles GS makes with these axes. Then

$$x = \xi \cos\alpha + \eta \cos\beta + \zeta \cos\gamma;$$

$$\therefore\ \Sigma mx^3 = \cos^3\alpha\, \Sigma m\xi^3 + 3\cos^2\alpha \cos\beta\, \Sigma m\xi^2\eta + \dots\dots$$

If the body be symmetrical about any set of rectangular axes meeting at G, we have $\Sigma m\xi^3 = 0$, $\Sigma m\xi^2\eta = 0$, &c. $= 0$, so that this next term in the expression for the potential vanishes altogether. Thus the error of the preceding expression for V is comparable to only the fraction $\left(\frac{l}{\rho}\right)^4$ of the potential. This is the case with the earth, the form and structure of which are very nearly symmetrical about the principal axes at its centre of gravity.

259. In the investigation of this value for the potential, S has been supposed to be at a very great distance. But the expression is also very nearly correct wherever the point S be situated, provided the body be an ellipsoid whose strata of equal density are concentric ellipsoids of small ellipticity.

To prove this, we may use a theorem in attractions due to Maclaurin, viz. The potentials of confocal ellipsoids at any external point are proportional to their masses. Let us first consider the case of a solid homogeneous ellipsoid. Describe an internal confocal ellipsoid of very small dimensions and let a', b', c' be its semi-axes. Then because the ellipticity is very small, we can take a', b', c' so small that S may be regarded as a distant point with regard to the internal ellipsoid. Hence the potential due to the internal ellipsoid is

$$V' = \frac{M'}{\rho} + \frac{A' + B' + C' - 3I'}{2\rho^3},$$

where accented letters have the same meaning relatively to the internal ellipsoid that unaccented letters have with regard to the given ellipsoid. The error made in this expression is of the order $\left(\frac{a'}{\rho}\right)^4 V'$.

Hence, by Maclaurin's theorem, the potential V of the given ellipsoid is

$$V = \frac{M}{\rho} + \frac{M}{M'}\,\frac{A' + B' + C' - 3I'}{2\rho^3},$$

and the error is of the order $\left(\frac{a'}{\rho}\right)^4 V$.

If a, b, c be the semi-axes of the given ellipsoid, we have

$$a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2 = \lambda^2;$$

$$\begin{aligned} \therefore\ A &= M\frac{b^2+c^2}{5} \\ &= M\left(\frac{b'^2+c'^2}{5} + \frac{2}{5}\lambda^2\right) \\ &= \frac{M}{M'}A' + \frac{2}{5}M\lambda^2. \end{aligned}$$

Similarly,

$$B = \frac{M}{M'}B' + \frac{2}{5}M\lambda^2, \qquad C = \frac{M}{M'}\ C' + \frac{2}{5}M\lambda^2.$$

Also if (α, β, γ) be the direction angles of the line GS with reference to the principal axes at G, we have

$$\begin{aligned} I &= A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma \\ &= \frac{M}{M'}I' + \frac{2}{5}M\lambda^2. \end{aligned}$$

Hence, substituting, we have

$$V = \frac{M}{\rho} + \frac{A+B+C-3I}{2\rho^3}.$$

If a, b, c be arranged in descending order of magnitude, we can by diminishing the size of the internal ellipsoid make c' as small as we please. In this case we have ultimately $a' = \sqrt{a^2 - c^2}$. Let ϵ be the ellipticity of the section containing a and c the greatest and least semi-axis. Then $a' = a\sqrt{2\epsilon}$, and the error of the above expression for V is of the order $4\left(\frac{a}{\rho}\right)^4 \epsilon^2 V$.

The theorem being true for any solid homogeneous ellipsoid is also true for any homogeneous shell bounded by concentric ellipsoids of small ellipticity. For the potential of such a shell may be found by subtracting the potentials of the bounding ellipsoids, $A+B+C$ by Art. 6 being independent of the directions of the axes.

Lastly, suppose the body to be an ellipsoid whose strata of equal density are concentric ellipsoids of small ellipticity, the external boundary being homogeneous. Then the proposition being true for each stratum, is also true for the whole body. See the *Cambridge and Dublin Mathematical Journal*, 1851.

260. The demonstration of the following theorems will serve as an exercise on the preceding articles.

(1) If the law of attraction had been $-\phi$ (dist.) instead of the inverse square, the potential of a body on any external point S would have been represented by $\Sigma m\phi_1(PS)$, where $\phi(\rho)$ is the differential coefficient of $\phi_1(\rho)$. In this case, by reasoning in the same way, we get

$$V = M\phi_1(\rho) + \phi'(\rho)\frac{A+B+C}{4} - \frac{\rho}{2}\frac{d}{d\rho}\left(\frac{\phi(\rho)}{\rho}\right)I,$$

where A, B, C and I have the same meanings as before.

If (x', y', z') be the coordinates of S referred to the principal axes at G, the moment of the attraction of S about the axis of y is $= \frac{1}{\rho}\frac{d}{d\rho}\frac{\phi(\rho)}{\rho} \cdot (C-A)\,x'z'$. See Art. 152.

(2) If the attraction of a body on every external point be the same as that of a single particle placed at some point, then the mass of the particle is equal to the mass of the body, the point is the centre of gravity, and unless the law of attraction be as the direct distance, every axis through the centre of gravity is a principal axis at the centre of gravity. See the *Quarterly Mathematical Journal*, 1857.

(3) If two bodies of the same mass have their centres of gravity coincident and their ellipsoids of gyration at their centres of gravity *confocal*, and if they attract any the same distant point according to the law of nature, their attractions are the same in direction and magnitude.

(4) Let an ellipsoid be described having its semiaxes a, b, c such that $M\frac{2}{3}a^2 = B + C - A + \lambda$, $M\frac{2}{3}b^2 = C + A - B + \lambda$,

$M\frac{2}{3}c^2 = A + B - C + \lambda$ where λ is at our disposal and may be any quantity positive or negative which does not make a, b, c imaginary. Let an indefinitely thin shell of mass M be constructed bounded by similar ellipsoids and having this ellipsoid for one bounding surface. Then the attractions of the given body and this shell on any distant external point are the same in direction and magnitude.

(5) The attraction of a body two of whose principal moments at the centre of gravity A and B are equal and greater than the third attracts a distant point as if its mass were equally distributed over a straight line $\sqrt{3\frac{A-C}{M}}$ placed perpendicular to the plane of A, B with its middle point at the centre of gravity. This proposition is accurately true if the body be an indefinitely thin shell bounded by similar prolate spheroids. In any case it is necessary that the equal moments A, B should be greater than the third moment of inertia C.

(6) Whatever be the relative magnitudes of the three principal moments of inertia, the attraction on a distant point is the same as if the mass was distributed over the focal conic of the ellipsoid described in (4) so that the density at any point P is proportional to $\frac{AB}{\sqrt{AP \cdot PB}}$ where AB is the diameter through P.

261. *To find the Force-function due to the attraction of any body on any other distant body.*

Let G, G' be the centres of gravity of the two bodies, and let $GG' = R$. Let A, B, C; A', B', C' be the principal moments of inertia of the two bodies at G and G' respectively; I, I' the moments of inertia about GG', and let M, M' be the masses of the two bodies.

Let m' be any element of the body M' situated at the point S, and let $GS = \rho$. Then the potential of the body M at m' is $m'\left\{\frac{M}{\rho} + \frac{A+B+C-3I_1}{2\rho^3}\right\}$, where I_1 is the moment of inertia of the body M about GS. We have now to sum this expression for all values of m'. This gives

$$M\Sigma\frac{m'}{\rho}+\Sigma m'\frac{A+B+C-3I_1}{2\rho^3}.$$

The first term by the same reasoning as before gives

$$\frac{MM'}{R}+M\frac{A'+B'+C'-3I'}{2R^3}.$$

In the second term, let x', y', z' be the co-ordinates of m' referred to G' as origin. Then

$$\rho=R\left(1+\frac{x'}{R}+\text{squares of } x', y', z'\right)$$

$$I_1=I\left(1+\alpha x'+\beta y'+\gamma z'+\text{squares}\right),$$

where α, β, γ are some constants. Substituting these, and remembering that $\Sigma m'x'=0$, $\Sigma m'y'=0$, $\Sigma m'z'=0$, we get

$$M'.\frac{A+B+C-3I}{2R^3}\left\{1+\begin{pmatrix}\text{terms depending on the}\\ \text{squares of } x', y', z'\end{pmatrix}\right\}.$$

Hence the required force-function is

$$V=\frac{MM'}{R}+M\frac{A'+B'+C'-3I'}{2R^3}+M'\frac{A+B+C-3I}{2R^3}.$$

The error of this expression is of the order $\left(\frac{ll'}{R^2}\right)^2 V$, where l, l' are any linear dimensions of the two bodies respectively.

262. *To find the moment of the attraction of the sun and moon about one of the principal axes of the earth at its centre of gravity.*

Let the principal axes of the earth at its centre of gravity be taken as the axes of reference, and let α, β, γ be the direction angles of the centre of gravity G' of the sun. Then if V be the potential of the sun or moon on the earth, we have

$$V=\frac{MM'}{R}+M\frac{A'+B'+C'-3I'}{2R^3}+M'\frac{A+B+C-3I}{2R^3},$$

where unaccented letters refer to the earth and accented letters to the sun or moon. Let θ be the angle the plane through the sun and the axis of y makes with the plane of xy, then $\frac{dV}{d\theta}$ is the required moment in the direction in which we must turn the body to increase θ. From the above expression, since θ enters only through I, we have

$$\frac{dV}{d\theta} = -\frac{3}{2}\frac{M'}{R^3}\frac{dI}{d\theta}.$$

Now $I = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma$, and by Spherical Trigonometry, we have

$$\left.\begin{aligned}\cos\gamma &= \sin\beta\sin\theta\\ \cos\alpha &= \sin\beta\cos\theta\end{aligned}\right\};$$

$$\therefore\ \frac{dI}{d\theta} = -2(A-C)\sin^2\beta\sin\theta\cos\theta;$$

$$\therefore\ \left.\begin{array}{l}\text{the moment required}\\ \text{about the axis of } y\end{array}\right\} = -3\frac{M'}{R^3}(C-A)\cos\alpha\cos\gamma.$$

In this expression the mass of the attracting body is measured in astronomical units. We may eliminate this unit in the following manner. Let n' be the mean angular velocity of the sun about the earth, R_0 its mean distance, so that if M be the mass of the earth, we have $\frac{M'+M}{R_0^{\,3}} = n'^2$. Now M is very small compared with M', so small that $\frac{M}{M'}$ is of the order of terms already neglected. Hence we may in the same terms put $\frac{M'}{R_0^{\,3}} = n'^2$, and therefore

$$\left.\begin{array}{l}\text{the moment of the sun's at-}\\ \text{traction about the axis of } y\end{array}\right\} = -3n'^2(C-A)\cos\alpha\cos\gamma\left(\frac{R_0}{R}\right)^3.$$

Let n'' be the mean angular velocity of the moon about the earth, so that, if M'' be the mass of the moon, R'_0 the mean distance, we have $\frac{M''+M}{R'^{\,3}_0} = n''^2$. Let ν be the ratio of the mass of the earth to that of the moon, then we have $\frac{M''(1+\nu)}{R'^3_0} = n''^2$, and therefore if R' be the distance of the moon

$$\left.\begin{array}{l}\text{the moment of the moon's}\\ \text{attraction about the axis of } y\end{array}\right\} = -\frac{3n''^2}{1+\nu}(C-A)\cos\alpha\cos\gamma\left(\frac{R'_0}{R'}\right)^3.$$

In the same way the moments about the other axes may be found. Putting κ for the coefficient, we have

$$\text{moment about axis of } x = -\kappa(B-C)\cos\beta\cos\gamma,$$
$$\text{moment about axis of } z = -\kappa(A-B)\cos a\cos\beta.$$

Motion of the Earth about its Centre of Gravity.

263. *To find the motion of the pole of the earth about its centre of gravity when disturbed by the attraction of the sun and moon, the figure of the earth being taken to be one of revolution.*

Let us consider the effect of these two bodies separately. Then, provided we neglect terms depending on the square of the disturbing force, we can by addition determine their joint effect.

The sun attracts the parts of the earth nearer to it with a force slightly greater than that with which it attracts the parts more remote, and thus produces a small couple which tends to turn the earth about an axis lying in the plane of the equator and perpendicular to the line joining the centre of the earth to the centre of the sun. It is the effect of this couple which we have now to determine. It clearly produces small angular velocities about axes perpendicular to the axis of figure. We shall also suppose that the initial axis of rotation so nearly coincided with the axis of figure, that we may regard the angular velocities about axes lying in the plane of the equator to be small compared with the angular velocity about the axis of figure.

In the first instance let us consider the orbit of the disturbing body to be fixed in space. This is very nearly true in the case of the sun, less nearly so for the moon. This limitation of the problem proposed will be found greatly to simplify the solution. We can now choose as our axes of reference in space, the normal to this fixed plane, and two other axes in the plane of motion.

Let us take as axes of reference in the earth, *GC* the axis figure, *GA*, *GB* moving in the earth so that the plane *AC* is always at right angles to the plane of the orbit of the disturbing body. Then following the usual notation, the equations of notation are by Art. 121,

$$\left.\begin{aligned} A\left(\frac{d\omega_1}{dt} - \omega_2\frac{d\chi}{dt}\right) - (A-C)\,\omega_2\omega_3 &= L \\ A\left(\frac{d\omega_2}{dt} + \omega_1\frac{d\chi}{dt}\right) + (A-C)\,\omega_1\omega_3 &= M \\ C\frac{d\omega_3}{dt} &= 0 \end{aligned}\right\}\ldots\ldots(\mathrm{I}).$$

The last of these equations shows that ω_3 is constant, let this constant value be denoted by n.

In the figure let *NS* be the plane of motion of the disturbing body, *Z* its pole, *S* the body moving in the direction *N* to *S*, and let its longitude measured *in the same direction from B*, the intersection of its plane of motion with the equator, be l. Let $\theta = ZC$, and let ψ be the angle ZC*

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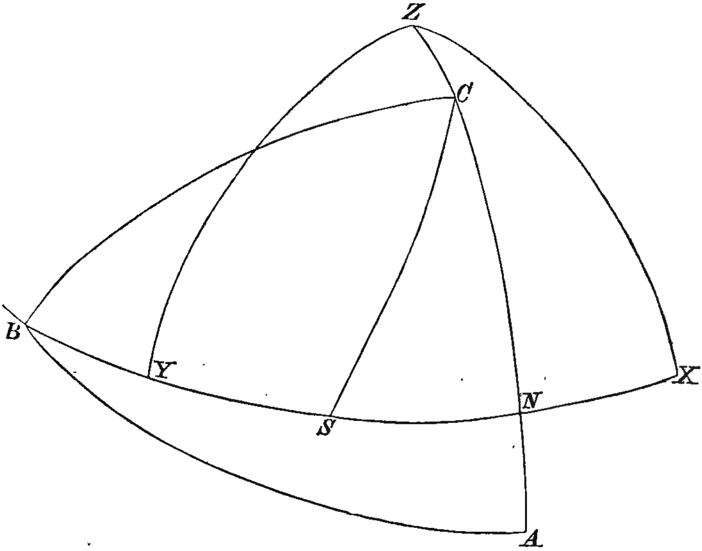
* We might also very conveniently have chosen as axes of reference in the body, *GC* the axis of figure, and axes *GA*, *GB* moving in the earth so that *GB* is the axis of the resultant couple produced by the action of the disturbing body on the earth. In this case $\frac{d\chi}{dt}$ is the rate at which a plane through the axis C and the disturbing body separates from a plane fixed in the earth. Hence $\frac{d\chi}{dt} = -n +$ small terms depending on the motion of the sun. For a first approximation, we may neglect these when multiplied by the small terms ω_1, ω_2. The equations (I) then become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + Cn\,\omega_2 &= 0, \\ A\frac{d\omega_2}{dt} - Cn\,\omega_1 &= -3n'^2(C-A)\left(\frac{R_0}{R}\right)^3\cos\alpha\cos\gamma \end{aligned}\right\}.$$



makes with some fixed great circle ZX. The geometrical equations will therefore be

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \omega_2 \\ -\sin\theta\frac{d\psi}{dt} &= \omega_1 \\ -\frac{d\chi}{dt}+\frac{d\psi}{dt}\cos\theta &= \omega_3 \end{aligned}\right\}\text{..................... (II).}$$

Since the plane AC moves so as always to contain the sun, we have $\alpha=\frac{\pi}{2}-\gamma$. Eliminating ω_2 we have therefore

$$\frac{d^2\omega_1}{dt^2}+\left(\frac{Cn}{A}\right)^2\omega_1=\frac{3n'^2}{2}\,\frac{C-A}{A}\cdot\frac{Cn}{A}\left(\frac{R_0}{R}\right)^3\sin 2\gamma.$$

Now γ is the angular distance of the sun from the axis of figure of the earth. Hence taking account of the motion of the sun, the right-hand side of this equation may be expressed in a series of the form

$$\Sigma P\sin(pt+Q),$$

It appears from the last of these equations that $\frac{d\chi}{dt}$ differs from $-\omega_3$ by a quantity of the order ω_1. Hence, as we shall neglect terms depending on the squares and products of the small quantities ω_1, ω_2 we may in the *small terms* put $\frac{d\chi}{dt} = -n$. The equations of motion now become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + Cn\omega_2 &= L \\ A\frac{d\omega_2}{dt} - Cn\,\omega_1 &= M \end{aligned}\right\}\text{...............(III)}.$$

The moments L and M have been found in Art. 262. We have, writing μ for the coefficient,

where p is small compared with n; being equal to 0 or n' where $\frac{n'}{n} = \frac{1}{365}$. Solving the equation we have

$$\omega_1 = \Sigma \frac{P}{\left(\frac{Cn}{A}\right)^2 - p^2} \sin(pt + Q).$$

Neglecting the small term p^2 in the denominator, we have

$$\omega_1 = \frac{3n'^2}{2n}\,\frac{C-A}{C}\left(\frac{R_0}{R}\right)^3 \sin 2\gamma + F\sin\left(\frac{Cn}{A}t + f\right),$$

where F and f are two arbitrary constants. Substituting in the equations of motion we have

$$\omega_2 = -F\cos\left(\frac{Cn}{A}t + f\right).$$

In the expression for ω_2 a term containing $\frac{d\gamma}{dt}$ has been omitted, because it is much smaller than the corresponding term in ω_1 since $\frac{d\gamma}{dt} = \frac{n}{365}$ nearly. The complementary function may be omitted for the reason given in Art. 264.

The motion of the axis C in space is therefore simply that due to an angular velocity ω_1 about the axis A. Since the plane AC moves so as always to contain the disturbing body, the axis of figure GC is at any instant moving perpendicular to the plane containing it and the disturbing body with an angular velocity equal to $\frac{3n'^2}{2n}\,\frac{C-A}{C}\left(\frac{R_0}{R}\right)^3 \sin 2\gamma$. If we resolve this in the directions along and perpendicular to ZC we easily find the values of $\frac{d\theta}{dt}$ and $\sin\theta\,\frac{d\psi}{dt}$.

$$\left.\begin{aligned} M &= -\mu \cos\alpha \cos\gamma \\ &= -\mu \sin\theta \cos\theta \cos^2 SN \\ &= -\mu \sin\theta \cos\theta \frac{1-\cos 2l}{2} \\ \text{Similarly} \qquad L &= -\mu \sin\theta \frac{\sin 2l}{2} \end{aligned}\right\} \ldots\ldots (\text{IV}).$$

To solve these equations, we must substitute for l its value in terms of t. Since μ is so small that its square is to be neglected, we may in *this* substitution suppose θ to be constant and that l is measured from a fixed point in space. By the theory of elliptic motion, we have

$$l = n't + \epsilon' + 2e' \sin(n't + \epsilon' - \zeta) + \ldots$$

where e', ϵ', ζ are respectively the eccentricity, epoch and longitude of the perigee of the orbit of the disturbing body. It will be seen that if the substitution be effected L and M will be expressed in a series of cosines and sines so that we may put

$$\left.\begin{aligned} L &= \Sigma F \cos(\lambda t + j) \\ M &= \Sigma G \sin(\lambda t + j) \end{aligned}\right\},$$

where λ is a multiple of n' and is therefore small.

To find ω_1, ω_2 we put

$$\left.\begin{aligned} \omega_1 &= \Sigma P \sin(\lambda t + j) \\ \omega_2 &= \Sigma Q \cos(\lambda t + j) \end{aligned}\right\},$$

then substituting we easily find

$$\left.\begin{aligned} AP\lambda + Cn\,Q &= F \\ AQ\lambda + Cn\,P &= -G \end{aligned}\right\}.$$

The most important terms in the values of L and M are those in which $\lambda = 0$ or $\lambda = 2n'$. In all the other terms F and G contain powers of e'. Since then $\frac{n'}{n} = \frac{1}{365}$ we may consider λ small. It is also clear that when we substitute in the geometrical equations the values of ω_1 ω_2 to find θ and ψ, no term will rise on integration in which λ is not small. We have therefore very nearly $Q = \frac{F}{Cn}$, $P = -\frac{G}{Cn}$, or substituting

$$\left.\begin{aligned} \omega_1 &= -\frac{M}{Cn} \\ \omega_2 &= \frac{L}{Cn} \end{aligned}\right\} \text{......................} \quad \text{(V).}$$

The motion therefore of the pole of the earth in space referred to the pole of the orbit of the disturbing body as origin is given by the equations

$$\left.\begin{aligned} \frac{d\theta}{dt} &= -\frac{\mu}{2Cn} \cdot \sin\theta \sin 2l \\ \frac{d\psi}{dt} &= -\frac{\mu}{2Cn} \cos\theta \, (1 - \cos 2l) \end{aligned}\right\} \text{.......} \quad \text{(VI).}$$

264. In this solution we have not yet considered the Complementary Functions. To find these we must solve

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + Cn\,\omega_2 &= 0 \\ A\frac{d\omega_2}{dt} - Cn\,\omega_1 &= 0 \end{aligned}\right\}.$$

We easily find

$$\left.\begin{aligned} \omega_1 &= H\sin\left(\frac{Cn}{A}t + K\right) \\ \omega_2 &= -H\cos\left(\frac{Cn}{A}t + K\right) \end{aligned}\right\}.$$

The quantities H and K depend on the initial values of ω_1 ω_2. As these initial values are unknown H and K must be determined by observation. If H had any sensible value it would be discovered by the variations produced by it in the position in space of the pole of the earth. The period of these would be $\frac{2\pi}{n}\frac{A}{C}$, as A and C are nearly equal in the case of the earth, this period is nearly equal to a day. No such inequalities have been found. If however any such inequality existed we might consider these two terms together as a separate inequality to be afterwards added to that produced by the other terms of $\omega_1\ \omega_2$ whose period is half a year.

The effect of the complementary function on the motion of the pole of the earth has been already considered. The motion is the same as if the earth were at any instant set in rotation about an axis whose direction-cosines are proportional to $H\sin\left(\frac{Cn}{A}t+K\right)$, $-H\cos\left(\frac{Cn}{A}t+K\right)$ and n and then left to itself. The instantaneous axis will describe a right cone of small angle round the axis of figure and also a right cone of small angle in space. Hence from this cause there can be no permanent change in the position in space of the axis of the earth. See Art. 212.

265. To find the position in space of the pole C of the earth at the end of any time t we must integrate equations (VI). Since we neglect the square of the disturbing force we may on the right-hand side consider θ constant, and l to be measured from a fixed point in space. By the theory of elliptic motion we have

$$R^2\frac{dl}{dt}=\text{constant}$$

$$=R_0^{\,2}n'\sqrt{1-e'^2}.$$

Now μ contains the factor $\left(\frac{R_0}{R}\right)^3$, let us then put

$$\frac{\mu}{2C}=Sn'^2\left(\frac{R_0}{R}\right)^3,$$

so that in the case in which the sun is the disturbing body $S=\frac{3}{2}\frac{C-A}{C}$, and in the case of the moon $S=\frac{3}{2}\frac{C-A}{C}\frac{1}{1+\nu}$.

The equations now become

$$\left.\begin{aligned}\frac{d\theta}{dl} &= -S\frac{n'}{n}\frac{R_0}{R\sqrt{1-e'^2}}\sin\theta\sin 2l\\ \frac{d\psi}{dl} &= -S\frac{n'}{n}\frac{R_0}{R\sqrt{1-e'^2}}\cos\theta(1-\cos 2l)\end{aligned}\right\}.$$

From the equation to the ellipse we have

$$\frac{R_0(1-e'^2)}{R} = 1 + e'\cos(l-L).$$

If this value of R be substituted in the equations it will be clear that all the terms which contain e' are periodic and do not rise on integration so as to become equally important with the others. Since then e' is small, being about equal to $\frac{1}{60}$, it will be needless to calculate these terms. We have therefore

$$\left.\begin{aligned}\theta &= \text{const.} + S\frac{n'}{2n}\sin\theta\cos 2l\\ \psi &= \text{const.} - S\frac{n'}{n}\cos\theta\left(l - \frac{\sin 2l}{2}\right)\end{aligned}\right\}.$$

266. Let us consider first the meaning of the term $-S\frac{n'}{n}\cos\theta\, l$ in the value of ψ. Let a point C_0 describe a small circle round Z the pole of the orbit of the disturbing planet, the distance CZ being constant and equal to the mean value of θ. Let the velocity be uniform and equal to $S\frac{n'^2}{n}\cos\theta$, and let the direction of motion be *opposite* to that of the disturbing body. Then C_0 represents the motion of the pole of the earth so far as this term is concerned. This uniform motion is called Precession.

Next let us consider the two terms

$$\left.\begin{aligned}\delta\theta &= S\frac{n'}{2n}\sin\theta\cos 2l\\ \delta\psi &= S\frac{n'}{2n}\cos\theta\sin 2l\end{aligned}\right\}.$$

If we put $x = \sin\theta\, d\psi$, $y = \delta\theta$, we have

$$\frac{x^2}{\left(S\frac{n'}{2n}\cos\theta\sin\theta\right)^2} + \frac{y^2}{\left(S\frac{n'}{2n}\sin\theta\right)^2} = 1,$$

which is the equation to an ellipse.

Let us then describe round C_0 as centre an ellipse whose semi-axes are $S\frac{n'}{2n}\cos\theta\sin\theta$ and $S\frac{n'}{2n}\sin\theta$ respectively perpendicular to and along ZC; and let a point C_1 describe this ellipse in a period equal to half the periodic time of the disturbing body. Also let the velocity of C_1 be the same as if it were a material point attracted by a centre of force in the centre varying as the distance. Then C_1 represents the motion of the pole of the earth as affected both by Precession and the principal parts of Nutation.

If we had chosen to include in our approximate values of θ and ψ any small term of higher order, we might have represented its effect by the motion of a point C_2 describing another small ellipse having C_1 for centre. And in a similar manner by drawing successive ellipses we could represent geometrically all the terms of θ and ψ.

267. In the case of the sun we have $S = \frac{3}{2}\frac{C-A}{C}$, so that the precession in one year is $\frac{3}{2}\frac{C-A}{C}\frac{n'}{n}\cos\theta\, 2\pi$. It is shown in treatises on the Figure of the Earth that there is reason to put $\frac{C-A}{C} = \cdot 0031$. Also we have $\frac{n'}{n} = \frac{1}{365}$, and $\theta = 23^\circ . 8'$. This gives a precession of about $15''\cdot 42$ per annum. Similarly the coefficients of Solar Nutation in ψ and θ are respectively found to be $1''\cdot 23$ and $0''\cdot 53$. If we supposed the moon's orbit to be fixed, we could find in a similar manner the motion of the pole produced by the moon referred to the pole of the moon's orbit. In this case $S = \frac{3}{2}\frac{C-A}{C}\frac{1}{1+\nu}$. The value of θ varies between the limits

$23^\circ \pm 5^\circ$. Putting $\frac{n'}{n} = \frac{1}{27}$, $\nu = 80$, $\theta = 23^\circ$, we find a precession in one year a little more than double that produced by the sun. But the coefficients of what would be the nutations are about one-sixth of those produced by the sun.

268. In the preceding investigation we have considered the orbit of the disturbing body to be fixed in space. If it be not so, we must take the plane CA perpendicular to its instantaneous position at the instant under consideration. In this case the geometrical equations (II) will require some alteration. Neglecting as before the term $\frac{d\psi}{dt}\cos\theta$, we shall now have $\frac{d\chi}{dt} = -n + \frac{dv}{dt}$, where v* is a small term depending on the motion of the plane of the orbit. This term will be very small compared with n, and hence we may still put $\frac{d\chi}{dt} = -n$ in the small terms. The dynamical equations will not therefore be materially altered. With regard to the other two geometrical equations, it is clear that ω_2, ω_1 will continue to express the resolved parts of the velocity of C in space along and perpendicular to the instantaneous position of ZC. All the change that will be necessary is therefore to refer these velocities as given by equations (VI) to axes fixed in space, and then by integration we shall find the motion of C. This is the course we shall pursue in the case of the moon.

269. *To give a general explanation of the manner in which the attraction of the Sun causes Precession and Nutation.*

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* The value of v may be found in the following manner. The orbit at any instant is turning about the radius vector of the planet as an instantaneous axis. Let $\frac{du}{dt}$ be this angular velocity which we shall suppose known. Let Z, Z'; B, B' be two successive positions of the pole of the orbit and the extremity of the axis of B respectively. Then $ZB = \frac{\pi}{2} = Z'B'$. Hence the projections of ZZ', BB', on ZB are equal. This gives, since ZB is at right angles to both CZ and SB, $B\hat{S}B' \sin BS = Z\hat{C}Z' \sin ZC$. Now the angle $ZCZ' =$ and the angle $BSB' = du$, hence $\frac{dv}{dt} \cdot \sin\theta = -\frac{du}{dt}\sin l$.

If a body be set in rotation about a fixed point O under the action of no forces, we know that the momenta of all the particles are together equivalent to a couple which we shall represent by G about an axis called the invariable line. Let T be the Vis Viva of the body. If a plane be drawn perpendicular to the axis of G at a distance $\frac{\sqrt{T}}{G}\epsilon^2$ from the fixed point, then the whole motion is represented by making the momental ellipsoid whose parameter is ϵ roll on this plane. In the case of the earth, the axis OI of instantaneous rotation so nearly coincides with OC the axis of figure that the fixed plane on which the ellipsoid rolls is very nearly a tangent plane at the extremity of the axis of figure. This is so very nearly the case that we shall neglect the *squares* of all small terms depending on the resolved part of the angular velocity about any axis of the earth perpendicular to the axis of figure.

Let us now consider how this motion is disturbed by the action of the sun. The sun attracts the parts of the earth nearer to it with a slightly greater force than it attracts those more remote. Hence when the sun is either north or south of the equator its attraction will produce a couple tending to turn the earth about that axis in the plane of the equator which is perpendicular to the line joining the centre of the earth to the sun. Let the magnitude of this couple be represented by α, and let us suppose that it acts impulsively at intervals of time dt.

At any one instant this couple will generate a new momentum αdt about the axis of the couple α. This has to be compounded with the existing momentum G, to form a resultant couple G'. If the axis of α were exactly perpendicular to that of G we should have $G' = \sqrt{G^2 + (\alpha dt)^2} = G$ ultimately.

Let θ be the angle that the axis of G makes with OC, then θ is a quantity of that order of small quantities whose square is to be neglected. Taking the case when OG, OC and the axis of α are in one plane, for this is the case in which G' will most differ from G, we have

$$G'^2 = (G\cos\theta)^2 + (G\sin\theta + \alpha dt)^2$$
$$= G^2 + 2G\alpha\sin\theta dt \qquad (1)$$

Then α and θ being of the same order of small quantities, the term $\alpha\sin\theta$ is to be neglected. Hence we have $G' = G$. But the axis of G is altered in space by an angle $\frac{\alpha dt}{G}$ in a plane passing through OG and the axis of α.

Next let us consider how the Vis Viva T is altered. If T' be the new Vis Viva we have

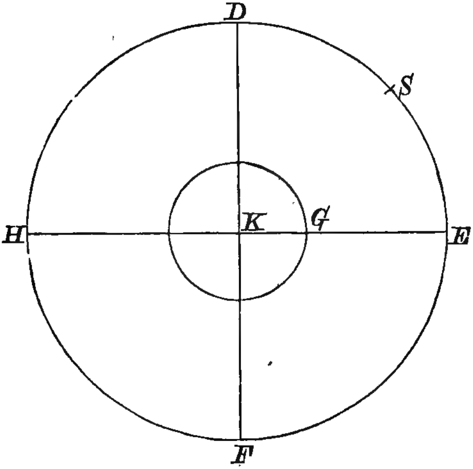
$$T' - T = 2 \text{ the work done by the couple } \alpha$$
$$= 2\alpha(\omega\cos\beta)\,dt \qquad (2)$$

where $\omega\cos\beta$ is the resolved part of the angular velocity about the axis of α. For the same reason as before the product of this angular velocity and α is to be neglected. Hence we have $T' = T$. It follows from these results that the distance $\frac{\sqrt{T}}{G}$ of the fixed plane from the fixed point is unaltered by the action of α.

Thus the fixed plane on which the ellipsoid rolls keeps at the same distance from the fixed point, so that the three lines OC, OI, OG being initially very near each other will always remain very close to each other. But the normal OG to this plane has a motion in space, hence the others must accompany it. This motion is what we call Precession and Nutation.

Lastly these small terms which have been neglected will not continually accumulate so as to produce any sensible effect. As the earth turns round in one day, the axis OC will describe a cone of small angle θ round OG. The axis about which the sun generates the angular velocity α is always at right angles to the plane containing the sun and OC. Hence, regarding the sun as fixed for a day, the angle θ in equation (1) changes its sign every half day. Thus G' is alternately greater and less than G. Similarly since the instantaneous axis describes a cone about OG it may be shown that T' is alternately greater and less than T.

270. Let us trace the motion of the axis OG through a whole year. Describe a sphere whose centre is at O and let us refer the motion to the surface of this sphere. Let K be the pole of the ecliptic and let the sun S describe the circle $DEFH$ of which K is the pole. Let DF be a great circle perpendicular to KG, then since OG and the axis of figure of the earth are so close that we may treat them as coincident, D and F will be the intersections of the equator and ecliptic. When the sun is north or south of the equator, its attraction generates the couple α, which will be positive or negative according as the sun is on one side or the other. This couple vanishes when the sun is passing through the equator at D or F. If the sun be anywhere in DEF, *i. e.* north of the equator, G is moved in a direction perpendicular to the arc GS towards D. If the sun be anywhere in FHD, α has the opposite sign and hence G is again moved perpendicular to the instantaneous position of GS but still towards D. Considering the whole effect produced in one year while the sun describes the circle $DEFH$, we see that G will be moved a very small space towards D, *i. e.* in the direction opposite to the sun's motion. Resolving this along the tangent to the circle centre K and radius KG, we see that the motion of



G is made up of (1) a uniform motion of *G* along this circle backwards, which is called Precession and (2) an inequality in this uniform motion which is one part of Solar Nutation. Again as the sun moves from *D* to *E*, *G* is moved inwards so that the distance *KG* is diminished, but as the sun moves from *E* to *F*, *KG* is as much increased. So that on the whole the distance *KG* is unaltered with an inequality which is the other part of Solar Nutation.

It is evident that each of these inequalities goes through its period in half a year.

271. *To explain the cause of Lunar Nutation.*

The attraction of the sun on the protuberant parts at the earth's equator causes the pole *C* of the earth to describe a small circle with uniform velocity round *K* the pole of the ecliptic with two inequalities, one in latitude and one in longitude, whose period is half a year. These two inequalities are called Solar Nutations. In the same way the attraction of the moon causes the pole of the earth to describe a small circle round *M* the pole of the lunar orbit with two inequalities. These inequalities are very small and of short period, viz. a fortnight, and are therefore generally neglected. All that is taken account of is the uniform motion of *C* round *M*. Now *K* is the origin of reference, hence if *M* were fixed the motion of *C* round *M* would be represented by a slow uniform motion of *C* round *K* together with two inequalities whose magnitude would be equal to the arc *MK* or 5 degrees and whose period would be very long, viz. equal to that of *C* round *K* produced by the uniform motion. But we know by Lunar Theory that *M* describes a circle round *K* as centre with a velocity much more rapid than that of *C*. Hence the motion of *C* will be represented by a slow uniform motion round *K* together with two inequalities which will be the smaller the greater the velocity of *M* round *K* and whose period will be nearly equal to that of *M* round *K*. This period we know to be about 19 years. These two inequalities are called Lunar Nutation. It will be perceived that their origin is different from that of Solar Nutation.

272. *To calculate the Lunar Precession and Nutation.*

Let K be the pole of the ecliptic, M that of the Lunar orbit, C the pole of the earth. Let KX be any fixed arc, $KC = \theta$, $XKC = \psi$, then we have to find θ and ψ in terms of t. By what precedes the velocity of C in space is at any instant in a direction perpendicular to MC, and equal to

$$-\frac{3n''^2}{2n}\,\frac{C-A}{C}\,\frac{1}{1+\nu}\cos MC\sin MC.$$

For the sake of brevity let the coefficient of $\cos MC \sin MC$ be represented by P. Then resolving this velocity along and perpendicular to KC, we have

$$\left.\begin{aligned}\frac{d\theta}{dt} &= -P\sin MC\cos MC\sin KCM\\ \sin\theta\frac{d\psi}{dt} &= -P\sin MC\cos MC\cos KCM\end{aligned}\right\}.$$

By Lunar theory we know that M regredes round K uniformly, the distance KM remaining unaltered. Let then $KM = i$, and the angle $XKM = -mt + \alpha$.

Now by spherical trigonometry,

$$\cos MC = \cos i\cos\theta + \sin i\sin\theta\cos MKC,$$

$$\begin{aligned}\sin MC\cos KCM &= \frac{\cos i - \cos MC\cos\theta}{\sin\theta}\\ &= \cos i\sin\theta - \sin i\cos\theta\cos MKC,\end{aligned}$$

$$\sin MC\,.\,\sin KCM = \sin i\sin MKC.$$

Substituting these, we have

$$\frac{d\theta}{dt} = -P\left\{\sin i\cos i\cos\theta\sin MKC + \frac{1}{2}\sin^2 i\sin\theta\sin 2MKC\right\};$$

$$\begin{aligned}\sin\theta\frac{d\psi}{dt} = -P\Big\{&\sin\theta\cos\theta\left(\cos^2 i - \frac{1}{2}\sin^2 i\right)\\ &- \sin i\cos i\cos 2\theta\cos MKC - \frac{1}{2}\sin^2 i\sin\theta\cos\theta\cos 2MKC\Big\}.\end{aligned}$$

For a first approximation we may neglect the variations of θ and ψ when multiplied by the small quantity P. Hence $\frac{d\theta}{dt}$ contains only periodic terms, and the inclination θ has no permanent alteration. But $\frac{d\psi}{dt}$ contains a term independent of MKC; considering only this term, we have

$$\psi = \text{constant} - P\cos\theta\left(\cos^2 i - \frac{1}{2}\sin^2 i\right)t.$$

This equation expresses the precessional motion of the pole due to the attraction of the moon. We may write this equation in the form

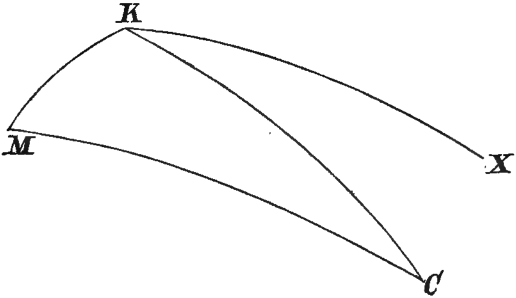
$$\psi = \psi_0 - pt.$$

To find the nutations, we must substitute for MKC its approximate value

$$MKC = (-m+p)t + \alpha - \psi_0.$$

We then have after integration

$$\theta = \text{const.} - \frac{P\sin i\cos i\cos\theta}{m-p}\cos MKC - \frac{P\sin^2 i\sin\theta}{4(m-p)}\cos 2MKC.$$



The second of these two periodic terms being about one-fiftieth part of the first, which is itself very small, is usually neglected. Also p is very small compared with m, hence we have

$$\theta = \theta_0 - \frac{P \sin i \cos i \cos \theta}{m} \cos MKC.$$

This term expresses the Lunar Nutation in the obliquity.

In the same way by integrating the expression for ψ, and neglecting the very small terms, we have

$$\psi = \psi_0 - P \cos\theta \left(\cos^2 i - \frac{1}{2}\sin^2 i\right) t - P\frac{\sin 2i}{2m} \cdot \frac{\cos 2\theta}{\sin\theta} \sin MKC.$$

The angle MKC is the longitude of the moon's ascending node, and the line of nodes is known to complete a revolution in about 18 years and 7 months. If we represent this period by T we have $MKC = -\frac{2\pi}{T} t + \text{constant}$.

Motion of the Moon about its centre of gravity.

273. *The centre of gravity of a rigid body describes an orbit which is nearly circular about a very distant centre of force, varying inversely as the square of the distance, and situated in one of the principal planes through the centre of gravity. It is required to determine the motion.*

Let GA, GB, GC be the principal axes at G, the centre of gravity of the body, and GC the axis perpendicular to the plane in which G moves. Let A, B, C be the moments of inertia about GA, GB, GC respectively, and let M be the mass of the body.

Let O be the centre of force, and let Ox be the initial line. Let $OG = r$, $GOx = \theta$. Let us suppose that the body turns round its axis OC in the same direction that the centre of gravity describes its orbit about O, and let the angle $OGA = \phi$.

Now if V be the potential due to the centre of force,

$$V = \frac{M}{r} + \frac{A + B + C - 3I}{2r^3},$$

and $$I = A\cos^2\phi + B\sin^2\phi.$$

Therefore the forces which act on the centre of gravity of the body are

$$\mu\frac{dV}{dr}=-\frac{\mu M}{r^2}\left\{1+\frac{3}{2}\frac{A+B+C-3I}{Mr^2}\right\}$$

along OG, and

$$-\mu\frac{dV}{rd\phi}=\frac{\mu M}{r^2}\cdot\frac{3}{2}\frac{B-A}{Mr^2}\sin 2\phi$$

perpendicular to OG, where μ is some constant depending on the intensity of the centre of force.

The moment of the forces tending to turn the body round G is

$$\mu\frac{dV}{d\phi}=-\frac{\mu M}{r}\,\frac{3}{2}\,\frac{B-A}{Mr^2}\sin 2\phi.$$

The equations of motion are therefore

$$\left.\begin{array}{l}\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2=-\dfrac{\mu}{r^2}\left\{1+\dfrac{3}{2}\dfrac{A+B+C-3I}{Mr^2}\right\}\\ \quad\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)=\dfrac{\mu}{r^2}\dfrac{3}{2}\dfrac{B-A}{Mr^2}\sin 2\phi\\ \dfrac{d^2\theta}{dt^2}+\dfrac{d^2\phi}{dt^2}=-\dfrac{\mu}{r^3}\dfrac{3}{2}\dfrac{B-A}{C}\sin 2\phi\end{array}\right\}.$$

It is given that the centre of gravity of the body describes an orbit which is nearly circular. Hence we may put

$$\left.\begin{array}{c}r=c\,(1+\rho)\\ \dfrac{d\theta}{dt}=n\left(1+\dfrac{d\psi}{dt}\right)\end{array}\right\},$$

where ρ and $\frac{d\psi}{dt}$ are small quantities whose squares may be neglected. We may also put $r=c$ in the *small* terms on the right-hand sides of these equations. Substituting and reducing, the equations become

$$\left.\begin{aligned}\frac{d^2\rho}{dt^2} - 3n^2\rho - 2n^2\frac{d\psi}{dt} &= -\frac{9}{4}n^2\gamma' - \frac{9}{4}n^2\gamma\cos 2\phi\\ 2\frac{d\rho}{dt} + \frac{d^2\psi}{dt^2} &= \frac{3}{2}n\gamma\sin 2\phi\\ \frac{d^2\phi}{dt^2} + n\frac{d^2\psi}{dt^2} &= -\frac{3}{2}\frac{B-A}{C}n^2\sin 2\phi\end{aligned}\right\},$$

where $\gamma = \frac{B-A}{Mc^2}$, $\gamma' = \frac{2C-A-B}{3Mc^2}$.

By hypothesis, the linear size of the body is small compared with its distance from the centre of force. Hence γ, γ' are both small quantities of at least the order of the square of the angular semi-diameter of the body, as seen from the centre of force. If A, B, C be nearly equal, as in the case of the earth or moon, γ, γ' will both be extremely small. We may therefore, in the first instance, neglect the terms containing γ and γ'*, and consider their effect afterwards. In this case,

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* We might also solve these equations without neglecting these terms in the following manner. Let $\rho = L\cos(pt+\alpha)+\ldots$ and $\frac{d\psi}{dt} = M\cos(pt+\alpha)+\ldots\ldots$ where L and M are small quantities whose squares are to be neglected. Substituting the equations become

$$\{L(p^2+3n^2)+2n^2M\}\cos(pt+\alpha)+\ldots = \frac{9}{4}\gamma' n^2 + \frac{9}{4}n^2\gamma\cos 2\phi,$$

$$\{2pL+pM\}\sin(pt+\alpha)+\ldots = -\frac{3}{2}n\gamma\sin 2\phi,$$

$$\frac{d^2\phi}{dt^2} - pnM\sin(pt+\alpha) - \ldots = -\frac{3}{2}\frac{B-A}{C}n^2\sin 2\phi.$$

Eliminating $\sin 2\phi$, the two last equations give

$$\frac{d^2\phi}{dt^2} = \left\{pnM + n\frac{B-A}{C}\cdot\frac{2pL+pM}{\gamma}\right\}\sin(pt+\alpha)+\ldots$$

$$\therefore\ \phi = p't + e + N\sin(pt+\alpha)+\ldots$$

$$\text{where } N = -\frac{n}{p}\left\{M + \frac{B-A}{C}\cdot\frac{2L+M}{\gamma}\right\}.$$

This value of ϕ must satisfy the two first equations. There are clearly two cases which may be considered, (1) when p' is not small so that we may put the two first equations which determine r and θ in terms of t, do not contain the disturbing force. Hence the motion of the

$\cos 2\phi = \cos 2(p't+\epsilon)$ and $\sin 2\phi = \sin 2(p't+\epsilon)$, (2) when p' and ϵ both vanish so that we have $\cos 2\phi = 1$ and $\sin 2\phi = 2N \sin(pt+\alpha)$.

In the first case, we find $p' = \frac{p}{2}$, $\epsilon = \frac{\alpha}{2}$ and the values of ρ, $\frac{d\psi}{dt}$, and ϕ are

$$\left.\begin{aligned} \rho &= L\cos(pt+\alpha) + L'\cos(nt+\alpha') + L''\cos\alpha'' \\ \frac{d\psi}{dt} &= M\cos(pt+\alpha) - 2L'\cos(nt+\alpha') + \frac{9}{8}\gamma' - \frac{3}{2}L''\cos\alpha'' \\ \phi &= \frac{1}{2}pt + \frac{1}{2}\alpha + N\sin(pt+\alpha) + 2L'\sin(nt+\alpha') \end{aligned}\right\}.$$

There are in these equations five arbitrary constants, viz. p, α, L', α' and $L''\cos\alpha''$ to be determined from the initial values of ρ, $\frac{d\rho}{dt}$, $\frac{d\psi}{dt}$, ϕ and $\frac{d\phi}{dt}$. The values of L, M, N are as follows:

$$L = \frac{3n^2(4n+3p)}{4p(p^2-n^2)}\gamma, \quad M = -\frac{3n(p^2+3np+3n^2)}{2p(p^2-n^2)}\gamma,$$

$$N = \frac{3n^2}{2p^2}\left\{\frac{p^2+3np+3n^2}{p^2-n^2}\gamma + \frac{B-A}{C}\right\}.$$

Since these must be small, we see that the solution fails if the initial conditions are such as to make p either small or nearly equal to n. In this case the angle ϕ makes a complete revolution in a period $\frac{4\pi}{p}$ and has two inequalities, one whose period is half that of ϕ, and another whose period is equal to that of the body round the centre of force. This latter we may call the elliptic inequality.

In the second case, we have the equation

$$p(p^2-n^2)(p^2-\beta^2) - 3n^2p(p^2+3n^2)\gamma = 0$$

to find p, where $\beta^2 = 3n^2\frac{B-A}{C}$. There are thus three terms in the expression for ϕ corresponding to the three roots. Let these roots be $p=0$, $p=p_1$, and $p=p_2$. The resulting values of ρ, $\frac{d\psi}{dt}$, and ϕ are then found to be

$$\left.\begin{aligned} \rho &= L_1\cos(p_1t+\alpha_1) + L_2\cos(p_2t+\alpha_2) + L_3 \\ \frac{d\psi}{dt} &= -\frac{p_1^2+3n^2}{2n^2}L_1\cos(p_1t+\alpha_1) - \frac{p_2^2+3n^2}{2n^2}L_2\cos(p_2t+\alpha_2) - \frac{3}{2}L_3 + \frac{9}{8}(\gamma+\gamma') \\ \phi &= \frac{p_1(p_1^2-n^2)}{6n^3\gamma}L_1\sin(p_1t+\alpha_1) + \frac{p_2(p_2^2-n^2)}{6n^3\gamma}L_2\sin(p_2t+\alpha_2) \end{aligned}\right\}.$$

There are in these equations five arbitrary constants L_1, α_1, L_2, α_2, and L_3. These are to be determined from the initial values of ρ, $\frac{d\rho}{dt}$, $\frac{d\psi}{dt}$, ϕ and $\frac{d\phi}{dt}$. The centre of gravity is the same as if the whole mass were collected into that point.

274. If other forces besides those due to the central force act on the body, the equations of motion will have to be modified. This is the case when the body considered is the moon. On the right-hand side of the first and second equations, we have forces due to the disturbing attractions of the sun and other bodies. These produce, as shewn in the Lunar Theory, terms in the expressions for r and $\frac{d\theta}{dt}$ of the form

$$r = c\{1 + L\cos(pt+\alpha) + \ldots\ldots\},$$

$$\frac{d\theta}{dt} = n + \beta t + Mn\cos(pt+\alpha) + \ldots\ldots$$

where β is some very small coefficient which causes a secular change of the moon's angular velocity.

275. If we substitute the value of $\frac{d\theta}{dt}$ in the third equation, we have the following equation to determine ϕ.

$$\frac{d^2\phi}{dt^2} = -n^2\frac{3}{2}\frac{B-A}{C}\sin 2\phi - \beta + npM\sin(pt+\alpha) + \ldots \text{(I)}$$

Now if ϕ be not small, the principal terms on the right-hand side of this equation will be the two first. Hence we may approximate to the value of ϕ by taking

$$\frac{d^2\phi}{dt^2} = -n^2\frac{3}{2}\frac{B-A}{C}\sin 2\phi - \beta.$$

Let us suppose that B is greater than A, so that ϕ is the angle the axis of *least* moment makes with OG, and for the

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investigation requires that the initial values of ϕ and $\frac{d\phi}{dt}$ should be small. In this case it appears that the body will always turn the same face to the centre of force and there will be two inequalities in the value of ϕ, one, which we may call the elliptic inequality, has nearly the same period as that of the body round the centre of force, the period of the other is very long.

sake of brevity put $n^2 \frac{3}{2} \frac{B-A}{C} = \frac{q^2}{2}$. Then integrating, we have

$$\left(\frac{d\phi}{dt}\right)^2 = H + \frac{q^2}{2} \cos 2\phi - 2\beta\phi,$$

where H is an arbitrary constant. To eliminate H, let it be known that at some instant (whether initially or not) $\phi = \epsilon$, $\frac{d\phi}{dt} = m$.

Then

$$m^2 = H + \frac{q^2}{2} \cos 2\epsilon - 2\beta\epsilon;$$

$$\therefore \left(\frac{d\phi}{dt}\right)^2 = m^2 + q^2 \sin^2 \epsilon + 2\beta\epsilon - q^2 \sin^2 \phi - 2\beta\phi$$

$$= L - q^2 \sin^2 \phi - 2\beta\phi,$$

where L has been written for the terms independent of ϕ. It is clear that the right-hand side of this equation must be positive throughout the motion. If the initial conditions be such that L is positive and small compared with q, it is clear that ϕ can never be great. In fact, β being very small compared with q, $\sin \phi$ must always be less than $\frac{\sqrt{L}}{q}$. In this case the principal axis GA will make small oscillations and its mean position will always very nearly point to the centre of force. To find these small oscillations we must recur to the equation (1). Since ϕ is small, we have

$$\frac{d^2\phi}{dt^2} + q^2\phi = -\beta + npM \sin (pt + \alpha) \ldots$$

$$\therefore \phi = H \sin (qt + K) - \frac{\beta}{q^2} + M \frac{np}{q^2 - p^2} \sin (pt + \alpha) + \ldots$$

If the initial conditions be such that $\frac{\sqrt{L}}{q}$ is not less than unity, there are no limits to the value of ϕ. If q be small, we have for a first approximation $\phi = mt + \epsilon$. For a second, we have

$$\phi = mt + \epsilon + \frac{q^2}{8m} \sin 2 (mt + \epsilon) - \frac{1}{2} \beta t^2 - M \frac{m}{p} \sin (pt + \alpha) + \dots$$

276. We may apply these two results to explain why the moon always turns the same face to the earth, while the earth does not always turn the same face to the sun. If we suppose the moon at any instant to be moving with its axis of least moment pointed towards the earth and its angular velocity about its axis of rotation to be nearly equal to that of the moon round the earth, then the above analysis shows that the axis of least moment will continue always to point very nearly to the earth. The mean angular velocity of the moon about its axis will immediately become equal to that of the moon round the earth and will partake of all its secular changes.

277. By comparing the value of ϕ obtained by theory with the results of observation, we may hope to obtain some indications of the value of q^2 and thence of $\frac{B-A}{C}$. If the term $H \sin (qt + K)$ could be detected by observation, we should deduce the value of $\frac{B-A}{C}$ from its period.

Among the other terms of the expression for ϕ those will be best suited to discover the value of q which have the largest coefficients, that is those in which either the numerator M is the greatest or the denominator $q^2 - p^2$ the least possible. By examining the numerical value of their coefficients Laplace has shown that if $\frac{B-A}{C}$ were as great as ·03 the elliptic inequality could be recognized by observation, and if it were between ·0014 and ·003 the annual equation could be observed.

278. The motion of a rigid body about a distant centre of force has been investigated on the supposition that the motion takes place entirely in one plane. On examining the equations written down in Art. 273, we see that they would be exactly satisfied if r were constant and $\phi = 0$. The case there-

fore in which the centre of gravity describes a circular orbit, and the rigid body always turns the axis of least moment towards the centre of force, is one of *steady motion*. The preceding investigation also shows that this motion is *stable* for all disturbances which do not alter the plane of motion. It remains now to determine the effect of these disturbances in the more general case when the motion takes place in three dimensions.

The whole attraction of the centre of force on the body is equivalent to a single force acting at the centre of gravity, and a couple. If the size of the body be small compared with its distance from the centre of force, we may neglect the effect of the motion of the body about its centre of gravity in modifying the resultant force. The motion of the centre of gravity will then be the same as if the whole were collected into a single particle. The problem is therefore reduced to the following. A rigid body turns about its centre of gravity G, and is acted on by a centre of force E which moves in a given manner. In the case in which the rigid body is the moon, this centre of force, i.e. the earth, moves in a nearly circular orbit in a plane which itself also has a slow motion in space. This motion is such that a normal GM to the instantaneous orbit describes a cone of small angle about a normal GK to the ecliptic. The two normals maintain a nearly constant inclination of about $5^0.8'$; and the motion of the normal to the instantaneous orbit is nearly uniform.

279. It will clearly be convenient to refer the motion to axes GX, GY, GZ fixed in space such that GZ is normal to the ecliptic. Let GA, GB, GC be the principal axes of the moon at the centre of gravity G. Let (p, q, r) be the direction-cosines of GZ referred to the co-ordinate axes GA, GB, GC. Then we have, since GZ is fixed in space

$$\left.\begin{aligned} \frac{dp}{dt} - \omega_3 q + \omega_2 r = 0 \\ \frac{dq}{dt} - \omega_1 r + \omega_3 p = 0 \\ \frac{dr}{dt} - \omega_2 p + \omega_1 q = 0 \end{aligned}\right\} \text{..................... (I).}$$

Now our object is to find the small oscillations about the state of steady motion in which GZ, GC, GM all coincide. We shall therefore have p, q, ω_1, ω_2 all small, and r very nearly equal to unity. The equations (I) will therefore become

$$\left.\begin{aligned} \frac{dp}{dt} - nq + \omega_2 = 0 \\ \frac{dq}{dt} - \omega_1 + np = 0 \end{aligned}\right\},$$

where n is the mean value of ω_3.

Let λ, μ, ν be the direction-cosines of the centre of force E as seen from G. Then we have by Euler's equations and Art. 262,

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\,\omega_2\omega_3 = -3n'^2(B-C)\,\mu\nu \\ B\frac{d\omega_2}{dt} - (C-A)\,\omega_3\omega_1 = -3n'^2(C-A)\,\nu\lambda \\ C\frac{d\omega_3}{dt} - (A-B)\,\omega_1\omega_2 = -3n'^2(A-B)\,\lambda\mu \end{aligned}\right\} \text{......} \text{ (II).}$$

In the case of steady motion, the rigid body always turns the axis (GA) of least moment towards the centre of force, and $\omega_3 = n'$. We have then both μ and ν small quantities, so that in the first equation we may neglect their product $\mu\nu$, and in the second equation we may put $\nu\lambda = \nu$. Also, we may put $\omega_3 = n = n'$ in the small terms.

If l be the latitude of the earth as seen from the moon, we have

$$\begin{aligned} \sin l = \cos ZE &= p\lambda + q\mu + r\nu \\ &= p + \nu \text{ nearly.} \end{aligned}$$

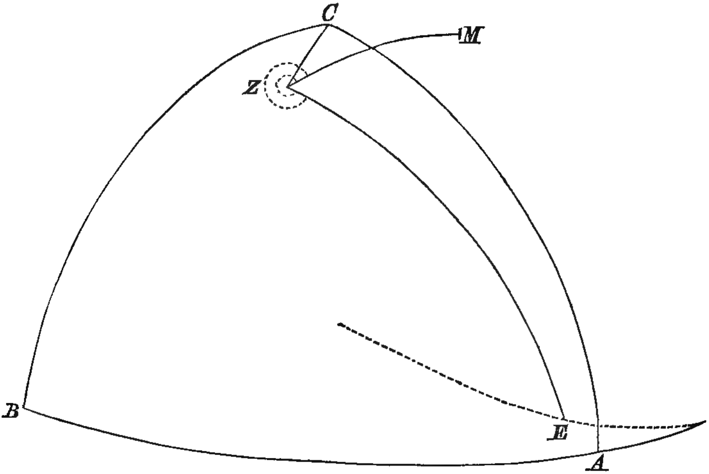
Hence the two first of Euler's equations take the form

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\,n\omega_2 = 0 \\ B\frac{d\omega_2}{dt} - (C-A)\,n\omega_1 = -3n^2(C-A)(-p+\sin l) \end{aligned}\right\} \text{....(III).}$$

If the earth, as seen from the moon, be supposed to move in a circular orbit in a plane making a constant inclination $\tan^{-1} k$ with the ecliptic, and the longitude of whose node is $-gt+\beta$, we shall have

$$\sin l = k \sin (n't + gt - \beta).$$

In this expression g measures the rate at which the node regredes, and is about the two hundred and fiftieth part of n. We shall therefore regard $\frac{g}{n}$ as a small quantity.



To solve these equations, it will be found convenient to substitute for ω_1, ω_2 their values in terms of p, q. We then have

$$\left.\begin{aligned}&A\frac{d^2q}{dt^2} + (A+B-C)\,n\frac{dp}{dt} - n^2(B-C)q = 0\\&B\frac{d^2p}{dt^2} - (A+B-C)\,n\frac{dq}{dt} + 4n^2(C-A)\,p = 3n^2(C-A)\sin l\end{aligned}\right\}.$$

To find p, q, let us put

$$\left.\begin{aligned}p &= P\sin\{(n'+g)\,t-\beta\}\\q &= Q\cos\{(n'+g)\,t-\beta\}\end{aligned}\right\},$$

where P, Q are some constants to be determined by substitution in the equation. We have

$$\left.\begin{aligned} Q\{A(n+g)^2+(B-C)n^2\}&=P(A+B-C)\,n\,(n+g) \\ P\{B(n+g)^2-4(C-A)n^2\}-Q(A+B-C)\,n\,(n+g)&=-3n^2k(C-A) \end{aligned}\right\}.$$

We may solve these equations to find P and Q accurately. In the case of the moon the ratios $\frac{A-B}{C}$, $\frac{B-C}{A}$, $\frac{C-A}{B}$ and $\frac{g}{n}$ are all small. If then we neglect the products of these small quantities, the first equation gives us $\frac{Q}{P}=1-\frac{g}{n}$. The second equation will then give

$$P=\frac{3nk\,(C-A)}{3n\,(C-A)-2Bg}.$$

As g is very small compared with n, we may regard P and Q as equal.

280. The complementary functions may be found in the usual manner by assuming

$$p=F\sin(st+H),$$
$$q=G\cos(st+H),$$

on substituting we have the quadratic

$$ABs^4-\{(A+B-C)^2-B(B-C)-4A(A-C)\}\,n^2s^2 \\ +4\,(A-C)\,(B-C)\,n^4=0,$$

to find s^2, and

$$\frac{G}{F}=\frac{(A+B-C)\,ns}{As^2+(B-C)n^2},$$

to find the ratio of the coefficients of corresponding terms in p and q. If the roots of this equation were negative p and q would be represented by exponential values of t, and thus they would in time cease to be small. It is therefore necessary for stability that the coefficient of s^2 should be negative and the product $(A-C)(B-C)$ positive. Both these conditions are probably satisfied in the case of the moon. For since $B-C$ and $A-C$ are both small, the term $(A+B-C)^2$ is much greater than the two other terms in the coefficient of s^2. Also, since the moon is flattened at its poles, we shall probably have A and B both less than C.

281. Let M be the pole of the moon's orbit, which is the same as that of the earth's orbit as seen from the centre of the moon. Then M is the pole of the dotted line in the figure of Art. 279. Therefore the angle EZM measured by turning ZE in the positive direction round Z until it comes into coincidence with ZM, is $=\frac{3\pi}{2}-\{(n+g)\,t-\beta\}$. Again, if the angle EZC be measured in the same direction, we have

$$\cos EZC=\frac{\cos EC-\cos CZ\cos ZE}{\sin CZ\sin ZE}$$

$$=\frac{\nu-r\,(p\lambda+q\mu+r\nu)}{\sqrt{p^2+q^2}\sin ZE}$$

$$=\frac{-p}{\sqrt{p^2+q^2}},\ \text{nearly.}$$

Hence we easily find

$$\sin EZC=\frac{-q}{\sqrt{p^2+q^2}}.$$

But $\quad \sin CZM=\sin EZM\cos EZC-\cos EZM\sin EZC$

$$=\frac{\cos\{(n+g)\,t+\beta\}\,p-\sin\{(n+g)\,t+\beta\}\,q}{\sqrt{p^2+q^2}}.$$

If now we substitute for p and q their values, it is clear that the terms in p and q, whose argument is $n+g$, disappear. So that if F and G were zero, the sine of the angle CZM would be absolutely zero. In this case the three poles C, Z, M must lie in an arc of a great circle, or which is the same thing, *the moon's equator, the moon's orbit, and the ecliptic must cut each other in the same line of nodes.*

If however F and G be not zero, but only very small, we have

$$\sin CZM = \frac{\Sigma F' \sin (s't + H')}{\sqrt{P^2 + \Sigma G'^2 \sin (s't + H')}},$$

where F', G' contain either F or G as a factor, and are therefore small. If then F and G be both small compared with P, the angle CZM will remain either always small or always nearly equal to π.

The intersection of the moon's equator with the ecliptic will then oscillate about the intersection of the moon's orbit with the ecliptic as its mean position. Since these oscillations are insensible, it follows that in the case of nature, the complementary functions must be extremely small compared with the terms depending directly on the disturbing force.

282. If we disregard the complementary functions we have $p = P \sin \phi$, $q = P \cos \phi$, where $\phi = (n' + g)\, t - \beta$. Now $\sin^2 CZ = p^2 + q^2$; therefore $CZ = -P$ very nearly. The value of CZ, the inclination of the lunar equator to the ecliptic, is known to be about 1°. 28′. Hence, since $\frac{g}{n} = \cdot 004$, we may deduce from the expression for P at the end of Art. 279 an approximation to the value of $\frac{C-A}{B}$. In this manner Laplace finds $\frac{C-A}{B} = \cdot 000599$.

EXAMPLES.

1. The attraction of any body of mass M on a distant particle may be found in the following manner. Let an indefinitely thin shell of mass $3M$ be constructed bounded by similar ellipsoids and having the ellipsoid of gyration at the centre of gravity for one bounding surface. Also let a particle of mass $4M$ be collected at the centre of gravity. Then the attraction of M on any distant particle is the same in direction and magnitude as if $4M$ attracted it and $3M$ repelled it.

2. Assuming (1) that the ellipticity of the earth's surface is so small that the expression for V in Art. 259 may be taken as true for external points near the surface of the earth, and (2) that at all points on the surface the resultant of the attraction and the centrifugal force is normal to the earth, prove that the potential at any point distant r from the centre is

$$V=\frac{M}{r}+\frac{Ma}{3r^3}\left(\epsilon-\frac{m}{2}\right)(1-3\sin^2\lambda),$$

where a is the semi-major axis of the earth, ϵ the ellipticity of any meridian, λ the latitude of the place, i.e. the angle the direction r makes with the axis-major, and m is the ratio of centrifugal force at the equator to equatoreal gravity.

Since the earth is a spheroid, the expression of Art. 259 becomes

$$V=\frac{M}{r}+\frac{C-A}{2r^3}(1-3\sin^2\lambda)\dots\dots\dots(1).$$

Since the resultant of the attraction and the centrifugal force is normal we have

$$\left(\frac{dV}{dx}+\omega^2x\right)dx+\left(\frac{dV}{dy}+\omega^2y\right)dy+\frac{dV}{dz}.dz=0\,;$$

$$\therefore\ V+\frac{\omega^2}{2}(x^2+y^2)=\text{const.}\ \dots\dots\dots(2).$$

Also since the earth is a spheroid of small ellipticity, we have along the surface

$$r=a(1-\epsilon\sin^2\lambda)\ \dots\dots\dots\dots\dots\dots\ (3).$$

Substitute from (1) and (3) in (2), and neglecting the squares of ϵ (see Art. 259), we should have an equation true for all values of λ. Equating to zero the coefficient of $\sin^2\lambda$, we get

$$\frac{M\epsilon}{a}-\frac{3}{2}\frac{C-A}{a^3}-\frac{\omega^2a^2}{2}=0.$$

Now $\omega^2a=m\frac{M}{a^2}$, substituting for ω we find $C-A=\frac{Ma^2}{3}(2\epsilon-m)$. If this be substituted in (1) we get the required value of V.

From this we may deduce Clairaut's theorem. Differentiating V with regard to r and adding the term $\omega^2r(1-\sin^2\lambda)$ arising from the centrifugal force resolved vertically, we get

$$g=\frac{M}{a^2}\left\{1+\epsilon-\frac{3}{2}m+\left(\frac{5}{2}m-\epsilon\right)\sin^2\lambda\right\}.$$

See *Cambridge and Dublin Math. Journal*, Vol. IV. 1851.

3. If the earth be formed of concentric spheroidal strata of small but different ellipticities and of different densities, shew that

$$\frac{C-A}{C}=\frac{\int\rho\,\frac{d\,(a^5\epsilon)}{da}\,da}{\int\rho\,\frac{da^5}{da}\,da},$$

where ϵ is the ellipticity and ρ the density of a stratum, the major-axis of which is a; the square of ϵ being neglected. It follows that if ϵ be constant, $\frac{C-A}{C}$ is independent of the law of density.

If we assume the law of density and the law of ellipticity given in the Figure of the Earth, this formula gives $\frac{C-A}{C}=\cdot 00313593$. See Pratt's *Figure of the Earth.*

4. If the earth were a homogeneous shell bounded by similar ellipsoids; the interior being empty, the precession would be the same as if the earth were solid throughout.

5. If the earth were a homogeneous shell bounded externally by a spheroid and internally by a concentric sphere, the interior being filled with a perfect fluid of the same density as the earth, show that the precession would be greater than if the earth were solid throughout.

Let (a, a, c) be the semi-axes of the spheroid, r the radius of the sphere. Then since the precession varies as $\frac{C-A}{C}$ by Art. 266, the precession is increased in the ratio $a^4c : a^4c-r^5$.

6. Show that the moon always turns the same face very nearly to that focus of her orbit in which the earth is not situated.

7. A particle m moves without pressure along a smooth circular wire of mass M with uniform velocity under the action of a central force situated in the centre of the wire attracting according to the law of nature. Show that this

system of motion is stable if $\frac{m}{M} > \frac{8+12\sqrt{6}}{25}$. The disturbance is supposed to be given to the particle or the wire, the centre of force remaining fixed in space.

8. A uniform ring of mass M and of very small section is loaded with a heavy particle of mass m at a point on its circumference, and the whole is in uniform motion about a centre of force attracting according to the law of nature. Show that the motion cannot be stable unless $\frac{m}{M+m}$ lies between ·815865 and ·8279.

This example shows (1) that if a ring, such as Saturn's ring, be in motion about a centre of force, its position cannot be stable, if the ring be uniform; and (2) that if, to render the motion stable, the ring be weighted, a most delicate adjustment of weights is necessary. A very small change in the distribution of the weights would change a stable combination to one that is unstable.

9. The centre of gravity of a body of mass M, symmetrical about the plane of xy, is G; and O is a point such that the resultant attraction of the body on O is along the line GO. Then if the body be placed with O coinciding with a fixed centre of force S, and be set in rotation about an axis through O perpendicular to the plane of xy with an angular velocity ω, G will, if undisturbed, revolve uniformly in a circle, always turning the same face towards O, provided $Ma\omega^2$ is equal to the resultant attraction along GO, where a is the distance GO. It is required to determine the conditions that this motion should be stable.

The motion being disturbed, O will no longer coincide with the centre of force S. Let two straight lines at right angles revolving uniformly round S as origin with an angular velocity ω be chosen as co-ordinate axes, and let x be initially parallel to OG. Let (x, y) be the co-ordinates of O, ϕ the angle OG makes with the axis of x, then x, y, ϕ are all small. Let V be the potential of the body at O, and let $\frac{d^2V}{dx^2} = \alpha$, $\frac{d^2V}{dx\,dy} = \gamma$, $\frac{d^2V}{dz^2} = \beta$. Let S be the amount of matter in the centre of force. Then the equations of motion of G, Art. 94, will reduce to

$$\left(\frac{d^2}{dt^2} - \omega^2 - \frac{S}{M}\alpha\right)x - \left(2\omega\frac{d}{dt} + \frac{S}{M}\gamma\right)y - 2\omega a\frac{d}{dt}\phi = 0,$$

$$\left(2\omega\frac{d}{dt} - \frac{S}{M}\gamma\right)x + \left(\frac{d^2}{dt^2} - \omega^2 - \frac{S}{M}\beta\right)y + a\frac{d^2}{dt^2}\phi = 0,$$

and the equation of angular momentum about S will lead to

$$2\omega ax + a\frac{d}{dt}y + (a^2+k^2)\frac{d}{dt}\phi = 0,$$

where k is the radius of gyration of the body about O. Combining these equations as a determinant and reducing we find that the differential equation in ξ, η, or ϕ is of the form

$$A\frac{d^4}{dt^4} + B\frac{d^2}{dt^2} + C = 0.$$

The condition of stability is that the roots of this equation should be real and negative. Hence A, B, C must be of the same sign and $B^2 > 4AC$. See Professor Maxwell's *Essay on Saturn's Rings*, page 69.

CHAPTER XII.

MOTION OF A FLEXIBLE BODY.

283. THE general term "Flexible Body" includes many other bodies besides strings. The motions treated of in these cases are generally small oscillations, and their discussion will properly form a subject by itself. The reader is therefore referred to any treatise on Sound and to the memoirs of Poisson, Cauchy and others on the subject. In the present chapter only the motion of a perfectly flexible string will be considered.

284. PROP. *To determine the general equations of motion of a string under the action of any forces.*

First. *Let the string be inextensible.*

Let Ox, Oy, Oz, be any axes fixed in space. Let $Xmds$, $Ymds$, $Zmds$, be the impressed forces that act on any element ds of the string whose mass is mds. Let u, v, w, be the resolved parts of the velocities of this element parallel to the axes. Then, by D'Alembert's principle, the element ds of the string is equilibrium under the action of the forces

$$mds\left(X-\frac{du}{dt}\right),\quad mds\left(Y-\frac{dv}{dt}\right),\quad mds\left(Z-\frac{dw}{dt}\right),$$

and the tensions at its two ends.

Let T be the tension at the point (x, y, z), then $T\dfrac{dx}{ds}$, $T\dfrac{dy}{ds}$, $T\dfrac{dz}{ds}$ are its resolved parts parallel to the axes. The resolved parts of the tensions at the other end of the element will be

$$T\frac{dx}{ds}+\frac{d}{ds}\left(T\frac{dx}{ds}\right)ds,$$

and two similar quantities with y and z written for x.

Hence the equations of motion are

$$\left.\begin{aligned} m\frac{du}{dt}&=\frac{d}{ds}\left(T\frac{dx}{ds}\right)+mX\\ m\frac{dv}{dt}&=\frac{d}{ds}\left(T\frac{dy}{ds}\right)+mY\\ m\frac{dw}{dt}&=\frac{d}{ds}\left(T\frac{dz}{ds}\right)+mZ \end{aligned}\right\}\text{..............(1).}$$

In these equations the variables s and t are independent. For any the same element of the string, s is always constant, and its path is traced out by variation of t. On the other hand, the curve in which the string hangs at any proposed time is given by variation of s, t being constant. In this investigation s is measured from any arbitrary point, fixed in the string, to the element under consideration.

To find the geometrical equations. We have

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1.$$

Differentiating this with respect to t, we get

$$\frac{dx}{ds}\frac{du}{ds}+\frac{dy}{ds}\frac{dv}{ds}+\frac{dz}{ds}\frac{dw}{ds}=0\text{ ....(2).}$$

The equations (1) and (2) are sufficient to determine x, y, z, and T, in terms of s and t.

285. These equations may be put under another form. Let ϕ, ψ, χ, be the angles made by the tangent at x, y, z, with the axes of co-ordinates. Then the equations (1) become

$$m\frac{du}{dt}=\frac{d}{ds}(T\cos\phi)+mX\text{.......... ......(3),}$$

with similar equations for v and w.

To find the geometrical equations, differentiate $\cos\phi = \frac{dx}{ds}$ with respect to t;

$$\therefore -\sin\phi\frac{d\phi}{dt} = \frac{du}{ds} \text{ .... } (4).$$

Similarly, by differentiating $\cos\psi = \frac{dy}{ds}$ and $\cos\chi = \frac{dz}{ds}$, we get two other similar equations for ψ and χ. Taking these six equations in conjunction with the following

$$\cos^2\phi + \cos^2\psi + \cos^2\chi = 1 \text{ .... } (5),$$

we have seven equations to determine u, v, w, ϕ, ψ, χ and T.

If the motion takes place in one plane, these reduce to the four following equations:

$$\left.\begin{aligned} m\frac{du}{dt} &= \frac{d}{ds}(T\cos\phi) + mX \\ m\frac{dv}{dt} &= \frac{d}{ds}(T\sin\phi) + mY \end{aligned}\right\} \text{ .... } (6),$$

$$\left.\begin{aligned} -\sin\phi\frac{d\phi}{dt} &= \frac{du}{ds} \\ \cos\phi\frac{d\phi}{dt} &= \frac{dv}{ds} \end{aligned}\right\} \text{ .... } (7).$$

The arbitrary constants and functions which enter into the solutions of these equations must be determined from the peculiar circumstances of each problem.

286. Secondly. *Let the string be elastic.*

Let σ be the unstretched length of the arc s, and let $md\sigma$ be the mass of an element $d\sigma$ of unstretched length or ds of stretched length. Then by the same reasoning as before, the equations of motion become

$$m\frac{du}{dt} = \frac{d}{d\sigma}\left(T\frac{dx}{ds}\right) + mX \text{ .... } (\text{i}),$$

and two similar equations for v and w. To find the geometrical equations we must differentiate

$$\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2 = \left(\frac{ds}{d\sigma}\right)^2,$$

the independent variables being now σ and t. Differentiating with regard to t we have

$$\frac{dx}{d\sigma}\frac{du}{d\sigma} + \frac{dy}{d\sigma}\frac{dv}{d\sigma} + \frac{dz}{d\sigma}\frac{dw}{d\sigma} = \frac{ds}{d\sigma}\frac{d}{dt}\left(\frac{ds}{d\sigma}\right).$$

But if λ be the modulus of elasticity of the string, we have

$$\frac{ds}{d\sigma} = 1 + \frac{T}{\lambda} \text{ .... (ii).}$$

Substituting we have

$$\frac{dx}{d\sigma}\frac{du}{d\sigma} + \frac{dy}{d\sigma}\frac{dv}{d\sigma} + \frac{dz}{d\sigma}\frac{dw}{d\sigma} = \left(1 + \frac{T}{\lambda}\right)\frac{1}{\lambda}\frac{dT}{dt} \text{ ...... (iii).}$$

These two equations (ii) and (iii) together with the three equations (i) will suffice for the determination of u, v, w, s and T in terms of σ and t.

If we wish to use the equations of motion in the forms corresponding to (3) or (6), the dynamical equations become

$$m\frac{du}{dt} = \frac{d}{d\sigma}(T\cos\phi) + mX,$$

with similar equations for v and w.

The geometrical equations corresponding to (4) or (7) may be found thus. We have

$$\frac{dx}{d\sigma} = \cos\phi\frac{ds}{d\sigma} = \cos\phi\left(1 + \frac{T}{\lambda}\right).$$

Differentiating, we have

$$\frac{du}{d\sigma} = -\sin\phi\frac{d\phi}{dt} + \frac{1}{\lambda}\frac{d}{dt}(T\cos\phi),$$

with similar expressions for v and w.

287. When the motion of the string takes place in one plane, it is often convenient to resolve the velocities along the tangent and normal to the curve.

Let u', v' be the resolved parts of the velocity of the element ds along the tangent and normal to the curve at that element. Let ϕ be the angle the tangent to the element ds makes with the axis of x. Then by Art. 94 or 120, the equations of motion are

$$\left.\begin{aligned} \frac{du'}{dt} - v'\frac{d\phi}{dt} &= X' + \frac{dT}{mds} \\ \frac{dv'}{dt} + u'\frac{d\phi}{dt} &= Y' + \frac{T}{m\rho} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

The geometrical equations may be obtained as follows. We have

$$u = u' \cos\phi - v' \sin\phi.$$

Differentiating with respect to s, we have by Art. 285,

$$-\frac{d\phi}{dt}\sin\phi = \left(\frac{du'}{ds} - \frac{v'}{\rho}\right)\cos\phi - \left(\frac{dv'}{ds} + \frac{u'}{\rho}\right)\sin\phi,$$

where ρ is the radius of curvature, and is equal to $\dfrac{ds}{d\phi}$. Since the axis of x is arbitrary in position, take it so that the tangent during its motion is parallel to it at the instant under consideration; then $\phi = 0$ and we have

$$0 = \frac{du'}{ds} - \frac{v'}{\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Similarly, by taking the axis of x parallel to the normal,

$$\frac{d\phi}{dt} = \frac{dv'}{ds} + \frac{u'}{\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

These four equations are sufficient to determine u', v', ϕ and T in terms of s and t.

288. The equations (2) and (3) may also be obtained in the following manner. The motion of the point P of the string being represented by velocities u' and v' along the tan-

gent PA and normal PO at P, the motion of a consecutive point Q will be represented by velocities $u'+du'$ and $v'+dv'$ along the tangent QB, and normal QO at Q. Let the arc $PQ=ds$, and let QN be a perpendicular on PA. Since the string is inextensible, the resultant velocity of Q resolved along the tangent at P must be ultimately the same as the resolved part of P in the same direction. Hence

$$(u'+du')\cos d\phi-(v'+dv')\sin d\phi=u',$$

or, proceeding to the limit,

$$du'-v'd\phi=0;$$

$$\therefore \frac{du'}{ds}-\frac{v'}{\rho}=0.$$

Again, $\frac{d\phi}{dt}$ is the angular velocity of PQ round P. Hence the difference of the velocities of P and Q resolved in any direction which is ultimately perpendicular to PQ must be equal to $PQ\frac{d\phi}{dt}$;

$$\therefore (u'+du')\sin d\phi+(v'+dv')\cos d\phi-v=ds\frac{d\phi}{dt},$$

or in the limit

$$\frac{d\phi}{dt}=\frac{dv'}{ds}+\frac{u'}{\rho}.$$

289. If the string be extensible, the dynamical equations become

$$\left.\begin{aligned}\frac{du'}{dt}-v'\frac{d\phi}{dt}&=X'+\frac{dT}{md\sigma}\\ \frac{dv'}{dt}+u'\frac{d\phi}{dt}&=Y'+\frac{T}{m\rho}\frac{ds}{d\sigma}\end{aligned}\right\}.$$

To find the geometrical equations, we may differentiate

$$u=u'\cos\phi-v'\sin\phi$$

with regard to σ. This gives by Art. 286

$$-\sin\phi\frac{d\phi}{dt}+\frac{1}{\lambda}\frac{d}{dt}(T\cos\phi)=\left(\frac{du'}{d\sigma}-\frac{v'}{\rho}\frac{ds}{d\sigma}\right)\cos\phi$$
$$-\left(\frac{dv'}{d\sigma}+\frac{u'}{\rho}\frac{ds}{d\sigma}\right)\sin\phi.$$

By the same reasoning as in Art. 287, this reduces to

$$\frac{1}{\lambda}\frac{dT}{dt}=\frac{du'}{d\sigma}-\frac{v'}{\rho}\left(1+\frac{T}{\lambda}\right),$$
$$\frac{d\phi}{dt}\left(1+\frac{T}{\lambda}\right)=\frac{dv'}{d\sigma}+\frac{u'}{\rho}\left(1+\frac{T}{\lambda}\right).$$

On Steady Motion.

290. Def. When the motion of a string is such that the curve which it forms in space is always equal, similar, and similarly situated to that which it formed in its initial position, that motion may be called steady.

291. Prop. *To investigate the steady motion of an inextensible string.*

It is obvious that every element of the string is animated with two velocities, one due to the motion of the curve in space, and the other to the motion of the string along the curve which it forms in space. Let a and b be the resolved parts along the axes of the velocity of the curve at the time t, and let c be the velocity of the string along its curve.

Then, following the usual notation, we have

$$\left.\begin{aligned}u&=a+c\,.\cos\phi\\ v&=b+c\,.\sin\phi\end{aligned}\right\}\text{.........................}(1).$$

Now a, b, c are functions of t only, hence

$$\frac{du}{ds}=-c\sin\phi\frac{d\phi}{ds}.$$

But $$\frac{du}{ds} = -\sin\phi \frac{d\phi}{dt} \text{ by Art. 285;}$$

$$\therefore \frac{d\phi}{dt} = c\frac{d\phi}{ds} \qquad (2).$$

Substituting the values of a and b in the equations of motion, Art. 284, we get

$$\left.\begin{aligned} \frac{da}{dt} + \frac{dc}{dt}\cos\phi - c\sin\phi\frac{d\phi}{dt} &= X + \frac{d}{ds}\left(\frac{T}{m}\cos\phi\right) \\ \frac{db}{dt} + \frac{dc}{dt}\sin\phi + c\cos\phi\frac{d\phi}{dt} &= Y + \frac{d}{ds}\left(\frac{T}{m}\sin\phi\right) \end{aligned}\right\}.$$

Substituting for $\frac{d\phi}{dt}$, these equations reduce to

$$\left.\begin{aligned} \frac{da}{dt} &= \left(X - \frac{dc}{dt}\cos\phi\right) + \frac{d}{ds}\left\{\left(\frac{T}{m} - c^2\right)\cos\phi\right\} \\ \frac{db}{dt} &= \left(Y - \frac{dc}{dt}\sin\phi\right) + \frac{d}{ds}\left\{\left(\frac{T}{m} - c^2\right)\sin\phi\right\} \end{aligned}\right\} \qquad (3).$$

The form of the curve is to be independent of t; hence, on eliminating T, the resulting equation must not contain t. This will not generally be the case unless $\frac{da}{dt}$, $\frac{db}{dt}$, $\frac{dc}{dt}$ are all constants. In any case their values will be determined by the known circumstances of the Problem. The above equations must then be solved, s being supposed to be the only independent variable, and t being constant.

292. If the string move uniformly in space, and the elements of the string glide uniformly along the string, $\frac{da}{dt} = 0$, $\frac{db}{dt} = 0$, $\frac{dc}{dt} = 0$. It will then follow from the above equations, that the form of the string will be the same as if it was at rest, but the tension will exceed the stationary tension by mc^2.

293. Ex. *Let an electric cable be deposited at the bottom of a sea of uniform depth from a ship moving with uniform*

velocity in a straight line, and let the cable be delivered with a velocity c *equal to that of the ship. Find the equation to the curve in which the string hangs.*

The motion may be considered *steady*, and the form of the curve of the string will be always the same.

If the friction of the water on the string be neglected, gravity diminished by the buoyancy of the water will be the only force acting on the string, let this be represented by g'. Hence the form of the travelling curve will be the common catenary, and the tension at any point will exceed the tension in the catenary by the weight of a length of string equal to $\frac{c^2}{g'}$.

Next let the friction of the water on any element of the cable be supposed to vary as the velocity of the element, and to act in a direction opposite to the direction of motion of the element*. Let μ be the coefficient of friction.

Let the axis of x be horizontal, and let x' be the abscissa of any point of the cable measured from the place where the cable touches the ground, in the direction of the ship's motion. Also let s' be the length of the curve measured from the same point. Then $x = x' + ct$, and $s = s' + ct$.

Following the same notation as before, we have

$$X = -\mu u, \qquad Y = -g' - \mu v.$$

But $$u = c - c\cos\phi, \qquad v = -c\sin\phi.$$

Hence the equations (3) become

$$\left.\begin{aligned} 0 &= -\mu c + \mu c\cos\phi + \frac{d}{ds}\left\{\left(\frac{T}{m} - c^2\right)\cos\phi\right\} \\ 0 &= -g' + \mu c\sin\phi + \frac{d}{ds}\left\{\left(\frac{T}{m} - c^2\right)\sin\phi\right\} \end{aligned}\right\}.$$

To integrate these put $\sin\phi = \frac{dy}{ds}$, $\cos\phi = \frac{dx}{ds}$.

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* Each element of the string has a motion both along the cable and transversely to it. The coefficients of these frictions are probably not the same, but they have been taken equal in the above investigation.

Hence,

$$\left.\begin{aligned} g'A &= -\mu cs + mcx + \left(\frac{T}{m} - c^2\right)\cos\phi \\ g'B &= -g's + \mu cy + \left(\frac{T}{m} - c^2\right)\sin\phi \end{aligned}\right\} \text{..............} (1),$$

where A and B are two arbitrary constants.

At the point where the cable meets the ground, we must have either $T=0$ or $\phi=0$. For if ϕ be not zero, the tangents at the extremities of an infinitely small portion of the string make a finite angle with each other. Then, if T be not zero, resolving the tensions at the two ends in any direction, we have an infinitely small mass acted on by a finite force. Hence the element will in that case alter its position with an infinite velocity. First, let us suppose that $\phi=0$. Also at the same point, $y=0$ and $s'=0$. Hence $B=-ct$.

Putting $\frac{\mu c}{g'}=e$, we get by division

$$\frac{dy}{dx'} = \frac{s'-ey}{A-ex'+es'} \text{........................} (2).$$

This is the differential equation to the curve in which the cable hangs.

To solve this equation*, find s' in terms of the other quantities,

$$s' = \frac{A\frac{dy}{dx'} - ex'\frac{dy}{dx'} + ey}{1 - e\frac{dy}{dx'}}.$$

Differentiating, we have

$$\sqrt{1+\left(\frac{dy}{dx'}\right)^2} = \frac{\frac{d^2y}{dx'^2}\cdot(A-ex'+e^2y)}{\left(1-e\frac{dy}{dx'}\right)^2}.$$

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* *Phil. Mag.* July 1858. The Astronomer Royal on the Mechanical Conditions of the deposit of a Submarine Cable.

Put p for $\frac{dy}{dx}$ where convenient, and put v for $A - ex' + e^2y$; the equation then becomes

$$\frac{1}{v}\frac{dv}{dx'} = \frac{-e\frac{dp}{dx'}}{(1-ep)\sqrt{1+p^2}},$$

in which the variables are separated, and the integrations can be effected. The equation can be integrated a second time, but the result is very long. The arbitrary constant A may have any value, depending on the length of the cable hanging from the ship at the time $t=0$.

The curve in its lower part resembles a circular arc or the lower part of a common catenary. But in its upper part the curve does not tend to become vertical, but tends to approach an asymptote making an angle $\cot^{-1}e$ with the horizon. The asymptote does not pass through the place of touching the ground but below it, its smallest distance being $\frac{A}{e\sqrt{e^2+1}}$, and it also passes below the ship.

If the conditions of the question be such that the tension at the lowest point of the cable is equal to nothing, the tangent to the curve at that point will not necessarily be horizontal. Let λ be the angle this tangent makes with the horizon. Referring to equations (1) we have when

$$x'=0,\ y=0,\ s'=0,\ T=0,\ \text{and}\ \phi=\lambda.$$

Hence $$A = -\frac{c^2}{g'}\cos\lambda,\ B = -\frac{c^2}{g'}\sin\lambda - ct.$$

The differential equation to the curve will now become

$$\frac{dy}{dx'} = \frac{-\frac{c^2}{g'}\sin\lambda + s' - ey}{-\frac{c^2}{g'}\cos\lambda + es' - ex'} \quad\text{..................(3)},$$

which can be integrated in the same manner as before. One case deserves notice; viz. when $e = \cot\lambda$. The equation is then evidently satisfied by $y = \frac{1}{e}x'$. The two constants in the integral of (3) are to be determined by the condition that when $x' = 0$, $y = 0$, then $\frac{dy}{dx'} = \tan\lambda$. Both these conditions are satisfied by the relation $y = \frac{1}{e}x'$. Hence this is the required integral. The form of the cable is therefore a straight line, inclined to the horizon at an angle $\lambda = \cot^{-1}e$; and the tension may be found from the formula

$$T = \frac{mg'y}{1 + \cos\lambda}.$$

On Initial Motions.

294. *A string, under the action of any forces in one plane, begins to move from a state of rest in the form of any given curve. To find the initial tension at any given point.*

Let Pds, Qds be the resolved parts of the forces respectively along the tangent and normal to any element ds. The force P is taken positively when it acts in the direction in which s is measured, and Q is positive when it acts in the direction in which ρ is measured along the normal inwards.

Let u, v be the velocities of the element along the tangent and normal. Then the equations of motion are by Art. 287

$$\frac{du}{dt} - v\frac{d\phi}{dt} = P + \frac{dT}{ds} \quad\text{.........................} \quad (1),$$

$$\frac{dv}{dt} + u\frac{d\phi}{dt} = Q + \frac{T}{\rho} \quad\text{.........................} \quad (2),$$

where T is the tension, ρ the radius of curvature, and ϕ the angle the tangent makes with any fixed straight line. The geometrical equations are

$$\frac{du}{ds} - \frac{v}{\rho} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\frac{dv}{ds} + \frac{u}{\rho} = \frac{d\phi}{dt} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Differentiating (1) and multiplying (2) by $\frac{1}{\rho}$, we get

$$\left.\begin{array}{r} \dfrac{d^2u}{ds\,dt} - v\dfrac{d^2\phi}{ds\,dt} - \dfrac{dv}{ds}\dfrac{d\phi}{dt} = \dfrac{dP}{ds} + \dfrac{d^2T}{ds^2} \\ \dfrac{1}{\rho}\dfrac{dv}{dt} + \dfrac{u}{\rho}\dfrac{d\phi}{dt} = \dfrac{Q}{\rho} + \dfrac{T}{\rho^2} \end{array}\right\} \ldots\ldots\ldots (5).$$

But by differentiating (3) we have, since $\frac{1}{\rho} = \frac{d\phi}{ds}$,

$$\frac{d^2u}{ds\,dt} - v\frac{d^2\phi}{ds\,dt} - \frac{1}{\rho}\frac{dv}{dt} = 0 \;\ldots\ldots\ldots\ldots\ldots (6).$$

Hence, subtracting the second of equations (5) from the first, we have by (4) and (6)

$$\frac{d^2T}{ds} - \frac{T}{\rho^2} + \frac{dP}{ds} - \frac{Q}{\rho} = -\left(\frac{d\phi}{dt}\right)^2.$$

In the beginning of the motion just after the string has been cut we may reject the squares of small quantities, hence $\left(\frac{d\phi}{dt}\right)^2$ may be rejected. Hence we have

$$\frac{d^2T}{ds^2} - \frac{T}{\rho^2} = -\frac{dP}{ds} + \frac{Q}{\rho} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

This is the general equation to determine the tension of a string just after it has been cut.

The two arbitrary constants introduced in the solution of this equation are to be determined by the circumstances of the case. If both ends of the string are free, we must have $T = 0$ at both ends.

Since the string begins to move from a state of rest we have initially $u=0$, $v=0$. At the end of a time dt, $\frac{du}{dt}dt$ and $\frac{dv}{dt}dt$ will be the velocities of any element of the string. Hence if ψ be the angle the initial direction of motion of any element of the string makes with the tangent to the element, we have by equations (1) and (2)

$$\tan\psi = \frac{Q+\frac{T}{\rho}}{P+\frac{dT}{ds}} \quad \text{.........................} \quad (8).$$

It must be remembered that the constants of integration are necessarily constant only throughout the length of the string at the time $t=0$. They may be functions of t and may be either continuous or discontinuous. For example, if a point of the string be absolutely fixed in space, the transverse action of the fixed point on the string may cause the constants to become discontinuous at that point. In this case equation (8) is not necessarily true in the immediate neighbourhood of the fixed point.

295. *A string is in equilibrium under the action of forces in one plane. Supposing the string to be cut at any given point, find the instantaneous change of tension.*

Let T_0 be the tension at any point (x, y) just before the string was cut. Then the forces P, Q satisfy the equations of equilibrium

$$0=P+\frac{dT_0}{ds},\quad 0=Q+\frac{T_0}{\rho}.$$

Hence

$$-\frac{dP}{ds}+\frac{Q}{\rho}=\frac{d^2T_0}{ds^2}-\frac{T_0}{\rho^2}.$$

If T' be the instantaneous change of tension, we have $T'=T-T_0$. The equation of the last article therefore becomes

$$\frac{d^2T'}{ds^2}-\frac{T'}{\rho^2}=0.$$

296. *A string rests on a smooth horizontal table and is acted on at one extremity by an impulsive tension, find the impulsive tension at any point and initial motion.*

Let T be the impulsive tension at any point P, $T+dT$ the tension at a consecutive point Q, then the element PQ is acted on by the tensions T and $T+dT$ at the extremities. Let ϕ be the angle the tangent at P to the string makes with any fixed line; u, v the initial velocities of the element resolved respectively along the tangent and normal at P to the string. Then, resolving along the tangent and normal, we have

$$\left.\begin{aligned} uds &= (T+dT)\cos d\phi - T \\ vds &= (T+dT)\sin d\phi \end{aligned}\right\};$$

therefore proceeding to the limit

$$u = \frac{dT}{ds}, \quad v = \frac{T}{\rho}.$$

But by Art. 288, we have $\frac{du}{ds} = \frac{v}{\rho}$. Hence the equation to find T becomes

$$\frac{d^2T}{ds^2} - \frac{T}{\rho^2} = 0.$$

This, as might have been expected from mechanical considerations, is the same as the equation in Art. 295.

297. The equation to find T cannot be integrated in any general manner. Several cases will be found discussed in Boole's *Differential Equations*, see Chapter XVII.

If $\rho = \frac{(s-a)(s-b)}{\beta}$ the integral can be shown to be

$$T = A(s-a)^m (s-b)^n + B(s-a)^n (s-b)^m,$$

where A and B are two arbitrary constants and (m, n) are the roots of the quadratic

$$x^2 - x - \left(\frac{\beta}{a-b}\right)^2 = 0.$$

See the *Cambridge Transactions*, Vol. VIII.

If $\rho = as^{\frac{2i}{2i\pm1}}$ the equation may be reduced to Riccati's equation by putting $T = e^{\int T' ds}$.

298. Ex. *A string is in equilibrium in the form of a circle about a centre of repulsive force in the centre. If the string be now cut at any point* A, *prove that the tension at any point* P *is instantaneously changed in the ratio of*

$$1 - \frac{e^{\pi-\theta} + e^{-(\pi-\theta)}}{e^{\pi} + e^{-\pi}} : 1,$$

where θ is the angle subtended at the centre by the arc AP.

Let F be the central force, then $P = 0$, and $Q = -F$. Let a be the radius of the circle. Then the equation becomes

$$\frac{d^2T}{ds^2} - \frac{T}{a^2} = -\frac{F}{a}.$$

Let s be measured from the point A towards P, then $s = a\theta$; also F is independent of s. Hence we have

$$T = Fa + A\epsilon^{\theta} + B\epsilon^{-\theta}.$$

To determine the arbitrary constants A and B we have the condition $T = 0$ when $\theta = 0$ and $\theta = 2\pi$;

$$\therefore\ T = Fa \cdot \left\{1 - \frac{\epsilon^{(\pi-\theta)} + \epsilon^{-(\pi-\theta)}}{\epsilon^{\pi} + \epsilon^{-\pi}}\right\}.$$

But just before the string was cut

$$T = Fa.$$

Hence the result given in the enunciation follows.

Small Oscillations.

299. Prop. *An inelastic string is suspended from two fixed points under the action of gravity so that it hangs in the form of a catenary, the parameter of which is* c. *Any small disturbance being given to the string in its own plane,*

it is required to determine the general equation to the small oscillations about the position of rest.

Let α be the angle the tangent at any point makes with the horizon when the string is at rest, and $\alpha+\phi$ the angle made by the tangent at the same point of the string at the time t.

Let u, v be the velocities of any element ds of the string resolved along the tangent and normal, and T' the tension of the element. Let the mass of a unit of length be taken as the unit. Then the general equations of motion of the string are by Art. 287,

$$\left.\begin{aligned}\frac{du}{dt}-v\frac{d\phi}{dt}&=-g\sin(\alpha+\phi)+\frac{dT'}{ds}\\ \frac{dv}{dt}+u\frac{d\phi}{dt}&=-g\cos(\alpha+\phi)+\frac{T'd(\alpha+\phi)}{ds}\end{aligned}\right\}.$$

Let the directrix of the catenary be taken as the axis of x, and let s be measured from that point of the string which coincides with the lowest point of the catenary when the string is in the position of equilibrium. Then the tension when the string is at rest is gy, which is equal to $\dfrac{gc}{\cos\alpha}$.

$$\text{Let } T'=\frac{gc}{\cos\alpha}+T. \quad \text{Also } \tan\alpha=\frac{s}{c};$$

$$\therefore \frac{d\alpha}{ds}=\frac{\cos^2\alpha}{c} \text{ and } \frac{dT'}{ds}=\frac{\cos^2\alpha}{c}\cdot\frac{dT'}{d\alpha}.$$

Substituting these values of T' and $\dfrac{dT'}{ds}$, and remembering that in small oscillations we may neglect the squares and products of the small quantities u, v, ϕ, we get

$$\frac{du}{dt}=-g\cos\alpha\,.\,\phi+\frac{\cos^2\alpha}{c}\frac{dT}{d\alpha}\quad\text{.......................}\quad(1),$$

$$\frac{dv}{dt}=g\sin\alpha\,.\,\phi+g\cos\alpha\,.\,\frac{d\phi}{d\alpha}+\frac{\cos^2\alpha}{c}\,.\,T\quad\text{........}\quad(2).$$

We have also by Art. 287 the two geometrical equations,

$$\left.\begin{aligned} \frac{du}{ds} - \frac{v}{\rho} &= 0 \\ \frac{dv}{ds} + \frac{u}{\rho} &= \frac{d\phi}{dt} \end{aligned}\right\},$$

where $\frac{1}{\rho} = \frac{d\alpha}{ds} + \frac{d\phi}{ds}$ is the reciprocal of the radius of curvature.

Changing the independent variable to α and neglecting the squares of small quantities, these reduce to

$$\left.\begin{aligned} v &= \frac{du}{d\alpha} \\ \frac{d^2u}{d\alpha^2} + u &= \frac{c}{\cos^2\alpha}\frac{d\phi}{dt} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots (3).$$

For the sake of brevity let us put u', v', ϕ'' for $\frac{du}{dt}$, $\frac{dv}{dt}$, $\frac{d^2\phi}{dt^2}$ respectively.

In order to solve these equations, we must eliminate T from the equations (1) and (2). Differentiating the second equation, we get

$$\frac{d^2u'}{d\alpha^2} = g\cos\alpha\,.\,\phi + g\cos\alpha\frac{d^2\phi}{d\alpha^2} + \frac{\cos^2\alpha}{c}\frac{dT}{d\alpha} - \frac{2\cos\alpha\sin\alpha}{c}T.$$

Subtracting equation (1) from this result, we have

$$\frac{d^2u'}{d\alpha^2} - u' = g\cos\alpha\left(\frac{d^2\phi}{d\alpha^2} + 2\phi\right) - \frac{2\cos\alpha\sin\alpha}{c}T.$$

Eliminating T from this by means of (2), we get

$$\cos\alpha\left(\frac{d^2u'}{d\alpha^2} + u'\right) + 2\left(\sin\alpha\frac{du'}{d\alpha} - u'\cos\alpha\right) = g\cos^2\alpha\frac{d^2\phi}{d\alpha^2}$$
$$+ 2g\sin\alpha\cos\alpha\frac{d\phi}{d\alpha} + 2g\phi \dots\dots\dots (4).$$

But by (3)

$$\frac{d^2u'}{d\alpha^2}+u'=\frac{c}{\cos^2\alpha}\phi'' \quad\dots\dots\dots\dots\dots\dots \quad (5);$$

$$\therefore\ \sin\alpha\frac{du'}{d\alpha}-u'\cos\alpha=c\int\frac{\sin\alpha}{\cos^2\alpha}\phi''d\alpha.$$

substituting these in (4), we get

$$\frac{c}{\cos\alpha}\phi''+2c\int\frac{\sin\alpha}{\cos^2\alpha}\phi''\,d\alpha=g\left(\cos^2\alpha\frac{d^2\phi}{d\alpha^2}+2\sin\alpha\cos\alpha\frac{d\phi}{d\alpha}+2\phi\right).$$

Differentiating again, we have

$$\frac{c}{\cos\alpha}\frac{d\phi''}{d\alpha}+\frac{3c\sin\alpha}{\cos^2\alpha}\phi''=g\cos^2\alpha\left(\frac{d^3\phi}{d\alpha^3}+4\frac{d\phi}{d\alpha}\right);$$

$$\therefore\ \frac{\cos^3\alpha\dfrac{d\phi''}{d\alpha}+3\cos^2\alpha\sin\alpha\phi''}{\cos^6\alpha}=\frac{g}{c}\left(\frac{d^3\phi}{d\alpha^3}+4\frac{d\phi}{d\alpha}\right);$$

integrating both sides, we have

$$\frac{\phi''}{\cos^3\alpha}=\frac{g}{c}\left\{\frac{d^2\phi}{d\alpha^2}+4\phi+f(t)\right\}.$$

Returning to the original notation, this may be written

$$\frac{d^2\phi}{dt^2}=\frac{g}{c}\cos^3\alpha\left\{\frac{d^2\phi}{d\alpha^2}+4\phi+f(t)\right\} \quad\dots\dots\dots\dots \quad (6).$$

This is the general equation to determine the small oscillations of a slack string.

Supposing the integration of this equation to have been effected, we have ϕ expressed in terms of α and t with two new arbitrary functions of α and t. These we may represent by $\psi(P)$ and $\chi(Q)$ where ψ and χ are arbitrary functions of two determinate combinations P and Q of the variables. The values of u and v may then be found by means of (3); the complementary functions introduced in this process will be of the form

$$\left.\begin{aligned}u&=A\cos\alpha+B\sin\alpha\\ v&=-A\sin\alpha+B\cos\alpha\end{aligned}\right\}.$$

where A and B are two more arbitrary functions of t. These correspond to a motion in space of the string without any change of form. Thus A and B are the resolved parts, parallel respectively to the directrix and axis of the catenary, of the common velocity of every element of the string. As the equation (6) has been obtained by differentiating (4) and finally integrating again, the arbitrary function $f(t)$ thus introduced may be determined in terms of A and B &c. by substituting in equation (4).

We have thus four arbitrary functions whose values have to be determined from the conditions of the question. Let α_0, α_1, be the values of α which correspond to the two fixed extremities of the string. Then the values of ϕ and $\frac{d\phi}{dt}$ are given by the question when $t=0$ for all values of α from $\alpha=\alpha_0$ to $\alpha=\alpha_1$; also the initial values of A and B are given. Thus the values of $\psi(P)$ and $\chi(Q)$ are determined for all values of P and Q between the two limits which correspond to $\alpha=\alpha_0$, $t=0$ and $\alpha=\alpha_1$, $t=0$. The forms of ψ and χ for values of P and Q exterior to these limits, and the values of A and B when t is not zero are to be found from the conditions that as t increases both u and v must vanish for all values of t when $\alpha=\alpha_0$ and $\alpha=\alpha_1$. It may therefore happen that the arbitrary functions A, B, ψ and χ are discontinuous. On the other hand the initial conditions may be such that these functions are all continuous.

300. The tension of the string will be given by equation (2), but another expression may also be found as follows.

Differentiating (2) and adding the result to (1), we obviously get by (5),

$$\frac{c}{\cos^2\alpha}\frac{d^2\phi}{dt^2}=g\cos\alpha\frac{d^2\phi}{d\alpha^2}+2\frac{\cos^2\alpha}{c}\frac{dT}{d\alpha}+\frac{T}{c}\frac{d\cos^2\alpha}{d\alpha},$$

$$\text{or }\cos\alpha\frac{dT}{d\alpha}-\sin\alpha\,.\,T=\frac{c^2}{2}\left(\frac{1}{\cos^3\alpha}\frac{d^2\phi}{dt^2}-\frac{g}{c}\frac{d^2\phi}{d\alpha^2}\right)$$

$$=\frac{cg}{2}\{4\phi+f(t)\};$$

$$\therefore \cos\alpha \, . \, T = \frac{cg}{2}\int\{4\phi + f(t)\}\,d\alpha\,;$$

$$\therefore \; T = \frac{cg}{2\cos\alpha}\{f(t)\,.\,\alpha + 4\int\phi\,d\alpha\}.$$

301. Supposing the value of ϕ to have been found from equation (6), it may be required to refer the motion to axes fixed in space. Let the directrix and axis of the catenary be taken as the axes of co-ordinates. Let $(x + \xi,\ y + \eta)$ be the co-ordinates of the point which when at rest was situated at the point (x, y), we have to find ξ and η in terms of ϕ. Let PQ be any element of the arc of the catenary which at the time t has been displaced into the position $P'Q'$. Let α be the angle PQ makes with the axis of x, then $\alpha + \phi$ is the angle $P'Q'$ makes with the same axis. Also dx and dy are the projections of PQ on the axes; $dx + \frac{d\xi}{ds}ds$, $dy + \frac{d\eta}{ds}ds$, are the corresponding projections of $P'Q'$;

$$\therefore \; dx + \frac{d\xi}{ds}ds = ds\cos(\alpha + \phi)$$

$$= ds\,(\cos\alpha - \sin\alpha\phi).$$

But $$dx = ds\cos\alpha,$$

and $$\sin\alpha = \frac{s}{y}\,;$$

$$\therefore \; \frac{d\xi}{ds} = -\frac{s}{y}\phi.$$

Similarly $$\frac{d\eta}{ds} = \frac{c}{y}\phi.$$

302. *A chain of length* 2l *is fastened at two points* A *and* B *in the same horizontal line whose distance apart is nearly equal to* 2l. *To find the small oscillations of the chain in a vertical plane.*

The motion of the chain may be deduced from the equation of Art. 299, but in order to show the different methods of proceeding, we shall begin by referring the motion to axes fixed in space.

When in equilibrium the chain will hang in the form of a catenary. Let C be the lowest point of the catenary and let the arc s be measured from C as origin. Let the directrix and axis of the catenary be taken as the axes of co-ordinates, the axis of x being horizontal. Let m be the mass of a unit of length of the chain, then the tension at any point of the chain whose ordinate is y, will be mgy.

When the chain is in motion let $(x+\xi,\ y+\eta)$ be the co-ordinates of the particle whose co-ordinates in equilibrium were (x, y) and let the tension T at that point be

$$T = mg\,(y+z).$$

The equations of motion of Art. 284 become

$$\begin{aligned}\frac{d^2\xi}{dt^2} &= g\,\frac{d}{ds}\left\{(y+z)\left(\frac{dx}{ds}+\frac{d\xi}{ds}\right)\right\}\\ &= g\,\frac{d}{ds}\left\{c + y\,\frac{d\xi}{ds} + z\,\frac{dx}{ds}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),\end{aligned}$$

for since the horizontal tension is constant and equal to mgc, we may put $mgy\,\dfrac{dx}{ds} = mgc$, also as in Art. 299 the terms which are of the order ξ^2, η^2, z^2 are neglected,

$$\begin{aligned}\frac{d^2\eta}{dt^2} &= g\,\frac{d}{ds}\left\{(y+z)\left(\frac{dy}{ds}+\frac{d\eta}{ds}\right)\right\} - g\\ &= g\,\frac{d}{ds}\left\{y\,\frac{d\eta}{ds} + z\,\frac{dy}{ds}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),\end{aligned}$$

by similar reasoning.

The geometrical equation is clearly

$$\left(dx + \frac{d\xi}{ds}\,dx\right)^2 + \left(dy + \frac{d\eta}{ds}\,ds\right)^2 = (dx)^2 + (dy)^2,$$

which, since $\dfrac{dy}{dx} = \dfrac{s}{c}$ reduces to

$$\frac{d\xi}{ds} = -\frac{s}{c}\,\frac{d\eta}{ds} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

To solve these equations, let us integrate equations (1) and (2) with regard to s and eliminate z; we have

$$\frac{d^2}{dt^2}\left\{\int \eta ds - \frac{s}{c}\int \xi ds\right\} = gy\left(\frac{d\eta}{ds} - \frac{s}{c}\frac{d\xi}{ds}\right) \ldots\ldots\ldots\ldots (4).$$

The distance between the two points of supports A and B is by the conditions of the problem not very different from the length of the chain. Hence the parameter c of the catenary is large. The second term on each side of this equation is therefore much smaller than the first. Further, we see from equation (3), that the horizontal displacement of any particle of the string is much smaller than η the vertical displacement. Thus the ratio of the second term to the first on each side of the equation is of the order $\left(\frac{s}{c}\right)^2$. The principal part of η will be found by neglecting these two terms. We may also put $y = c$.

We must however recollect that the integrations with regard to s will introduce arbitrary functions of t. These are not all of the order $\left(\frac{s}{c}\right)^2$ and they may therefore have to be retained. Let us put

$$\xi = -\int \frac{s}{c}\frac{d\eta}{ds}\, ds + A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5);$$

$$\therefore \int \xi ds = -\iint \frac{s}{c}\frac{d\eta}{ds}(ds)^2 + As + Bc \ldots\ldots\ldots\ldots\ldots (6),$$

where A and B are two arbitrary funtions of t to be determined by the conditions of the question. The equation (4) will now become

$$\frac{d^2}{dt^2}\left\{\int \eta ds - A\frac{s^2}{c} - Bs\right\} = gc\frac{d^2}{ds^2}\left\{\int \eta ds - A\frac{s^2}{c} - Bs\right\} + 2gA \ldots (7).$$

The integral of this equation is known to be

$$\int \eta ds - A\frac{s^2}{c} - Bs = 2gE + h\frac{s^2}{c} + ks$$
$$+ f(\sqrt{gc}\,t - s) + F(\sqrt{gc}\,t + s) \ldots\ldots\ldots (8),$$

where E is a function of t, and h, k are two constants such that

$$\frac{d^2E}{dt^2} = A + h.$$

The arbitrary functions f and F must be determined from the conditions of the question and may be either continuous or discontinuous. Let us now suppose that the motion of the chain is such that we may replace the functions by a finite number of trigonometrical terms of the form

$$L \sin m\left(\sqrt{gc}\, t \pm s\right).$$

Let $m\sqrt{gc} = n$, we have then

$$\eta = 2(A+h)\frac{s}{c} + B + k + \Sigma\left\{L \sin(nt + ms + \beta) + M \sin(nt - ms + \gamma)\right\},$$

$$\begin{aligned}\xi = -(A+h)\frac{s^2}{c^2} + A \\ -\Sigma\Bigg\{L\frac{s}{c}\sin(nt+ms+\beta) + \frac{L}{cm}\cos(nt+ms+\beta) \\ + M\frac{s}{c}\sin(nt-ms+\gamma) - \frac{M}{cm}\cos(nt-ms+\gamma)\Bigg\} \dots (9),\end{aligned}$$

where Σ implies summation for all values of m, and L, M are any constants which may be different in every term*.

If both the extremities of the string be fixed the values of ξ and η are wholly periodical and in this case $h = 0$. The functions A and B will be of the forms $A_0 \sin(nt + \beta)$ and

---

* The physical meaning of these terms is as follows. Suppose the string to receive a small initial displacement σ along its length so that the extremities A and B are moved along the tangents to the string at A and B respectively. Then the displacements of any point (x, y) are

$$\left.\begin{aligned}\xi &= \sigma \cos \alpha = \sigma\left(1 - \frac{s^2}{2c^2}\right) \\ \eta &= \sigma \sin \alpha = \sigma\frac{s}{c}\end{aligned}\right\}.$$

The constant terms in (ξ, η) express a transference of the whole catenary parallel to itself without change of form.

$B_0 \sin(nt+\beta)$, where A_0, B_0 are absolute constants. To determine the relations between the constants A_0, B_0, L, and M we have the conditions $\xi=0$, $\eta=0$ when $s=\pm l$. If we take the difference between the two equations deduced from the value of ξ by putting $s=\pm l$ we find

$$\{L\sin(nt+\beta)+M\sin(nt+\gamma)\}(ml\cos ml-\sin ml)=0.$$

One of these two factors must therefore vanish for all values of t. If the first factor be equated to zero we have $\gamma=\beta$ and $M=-L$. If the second factor be equated to zero, the possible values of m must satisfy the equation $\tan ml=ml$. Let us consider these cases in order.

First. If $\gamma=\beta$ and $M=-L$, we find since η must vanish when $s=\pm l$

$$\left.\begin{array}{l} A=-\dfrac{c}{l}L\sin ml.\cos(nt+\beta) \\ B=0 \end{array}\right\},$$

and since ξ must vanish when $s=\pm l$ we have very nearly

$$A=\frac{2}{cm}L(lm\sin ml+\cos ml)\cos(nt+\beta).$$

It will be sufficient if these two values of A agree when we neglect all small terms which contain the factor $\left(\dfrac{l}{c}\right)^2$. This gives $ml\tan ml=-2\left(\dfrac{l}{c}\right)^2$ nearly, and $\therefore ml=i\pi$ nearly, where i is any integer. The values of η and ξ will then be

$$\left.\begin{array}{l} \eta=\Sigma 2L\sin ms\cos(nt+\beta) \\ \xi=\Sigma\dfrac{2L}{mc}(\cos ml-\cos ms-ms\sin ms)\cos(nt+\beta) \end{array}\right\}\ldots\ldots(10),$$

where m has any value given by the equation $m=\dfrac{i\pi}{l}$.

Secondly. Let m have any value given by the equation $\tan ml=ml$. We find in the same way that the two values of A cannot in general be made to agree unless $L=M$, and

$\gamma = \beta$. We then have $A = 0$. The values of η and ξ will then be

$$\left.\begin{aligned}\eta &= \Sigma 2L(\cos ms - \cos ml)\sin(nt+\beta)\\ \xi &= \Sigma\frac{2L}{mc}(\sin ms - ms\cos ms)\sin(nt+\beta)\end{aligned}\right\}\text{......} (11),$$

where m has any value given by the equation $\tan ml = ml$.

The roots of the equation $\tan ml = ml$ may be found by continued approximation. The first is zero, but since m occurs in the denominator of some of the small terms in the investigation, this value is inadmissible. The others may be expressed by the formula $ml = (2i+1)\frac{\pi}{2} - \theta$, where θ is not very large. This gives $\frac{2\pi}{n}$, which is the time of vibration of the chain, equal to $\frac{4}{2i+1}\cdot\frac{l}{\sqrt{gc}}$ nearly. Thus the times of vibration of a chain slightly disturbed and left to itself are all short. The reader who may wish to see another method of discussing the small oscillations of a suspension chain may consult a memoir by Mr Röhrs in the ninth volume of the *Cambridge Transactions*.

303. In this discussion we have only considered the principal terms of η and ξ. But having found these, we may proceed to a second approximation, by substituting the values just found in the small terms of equation (4).

304. As an example of the mode of treating the oscillation of elastic strings, the following has been selected.

An elastic string whose weight may be neglected and whose unstretched length is l *has its extremities fixed at two points whose distance apart is* l′. *The string being very slightly disturbed from its position of rest, find the motion.*

Let A be one of the fixed points, and let AB be the string when unstretched and placed in a straight line. Let the extremity B be pulled until it reaches the other fixed point B'.

Let PQ be any element of the unstretched string, $P'Q'$ the same element at any time t. Let $AP=x$, and let the co-ordinates of P' be (x', y', z'). Let T and $T+dT$ be the tensions at P', Q'. Let M be the mass of the string, then $\frac{M}{l}dx$ is the mass of the portion dx. Hence the equations of motion are

$$\left.\begin{aligned}\frac{M}{l}\frac{d^2x'}{dt^2}&=\frac{d}{dx}\left(T\frac{dx'}{ds'}\right)\\ \frac{M}{l}\frac{d^2y'}{dt^2}&=\frac{d}{dx}\left(T\frac{dy'}{ds'}\right)\\ \frac{M}{l}\frac{d^2z'}{dt^2}&=\frac{d}{ds}\left(T\frac{dz'}{ds'}\right)\end{aligned}\right\},$$

where ds' is the length of the element $P'Q'$.

Now if E be the modulus of elasticity we have

$$\frac{ds'}{dx}=1+\frac{T}{E},$$

and $$\left(\frac{ds'}{dx}\right)^2=\left(\frac{dx'}{dx}\right)^2+\left(\frac{dy'}{dx}\right)^2+\left(\frac{dz'}{dx}\right)^2.$$

These five equations are sufficient to determine the motion.

Since the disturbance is very small, $\frac{dy'}{ds'}$, $\frac{dz'}{ds'}$ are very small, and $\frac{dx'}{ds'}$ is nearly equal to unity. Hence the first equation takes the form

$$\frac{M}{l}\frac{d^2x'}{dt^2}=\frac{dT}{dx},$$

and the fifth equation becomes

$$\frac{dx'}{dx}=1+\frac{T}{E}.$$

$$\therefore \frac{d^2x'}{dt^2}=\frac{El}{M}\frac{d^2x'}{dx^2}.$$

If we put $a^2 = \frac{El}{M}$, the integral of this equation is

$$x' = f(at - x) + F(at + x),$$

where f and F are two arbitrary functions. The discussion of these equations may be found in any treatise on Sound. The result is that each of these functions represents a wave which travels with the same velocity, viz. a. In the case therefore of the string, the motion will be represented by a series of waves travelling both ways along the string with the same velocity. This velocity is such that the time of traversing a length l of unstretched string or a length l' of stretched string is $\sqrt{\frac{Ml}{E}}$.

Suppose that when the string is suspended vertically from A, $M'g$ is the weight which would stretch the string to twice its length, then $E = M'g$. The time of traversing the length l of unstretched string is therefore $\sqrt{\frac{Ml}{M'g}}$.

In the second equation we may regard T as constant, its small variations being multiplied by the small quantity $\frac{dy'}{ds'}$.

Hence we can put

$$T = E\frac{l' - l}{l};$$

$$\therefore \frac{dy'}{ds'} = \frac{dy'}{dx}\frac{dx}{ds'}$$

$$= \frac{1}{1 + \frac{T}{E}} \cdot \frac{dy'}{dx}$$

$$= \frac{l}{l'}\frac{dy'}{dx}.$$

Hence the equation becomes

$$\frac{d^2y'}{dt^2}=\frac{Tl^2}{Ml'}\frac{d^2y'}{dx^2}.$$

The velocity of propagation along the unstretched string is therefore $\sqrt{\frac{Tl^2}{Ml'}}$. The time of traversing a length l of unstretched string or l' of stretched string is $\sqrt{\frac{l'M}{T}}$. Let us suppose the string fixed at A and hanging over a pulley at B', with a weight M'' attached to it. Then T being the tension in equilibrium, we have $T=M''g$. Hence the time which a wave takes to traverse the length of the string from A to B' is $\sqrt{\frac{l'M}{gM''}}$.

The question of the motion of a tight string is best considered in connection with the theory of sound. A discussion of this question may be found in nearly every treatise on that subject. For this reason it has been considered unnecessary to give more than this one proposition.

EXAMPLES.

1. An elastic ring without weight whose length when unstretched is given, is stretched round a circular cylinder. The cylinder is suddenly annihilated, show that the time which the ring will take to collapse to its natural length is $\sqrt{\frac{Ma\pi}{8\lambda}}$, where M is the mass of the string, λ its modulus of elasticity, and a is the natural radius.

2. A homogeneous light inextensible string is attached at its extremities to two fixed points, and turns about the straight line joining those points with uniform angular velocity. Find the form of the string, supposing its figure permanent.

Result. Let the straight line joining the fixed points be the axis of x, then the form of the string is a plane curve whose equation is

$$1+\left(\frac{dy}{dx}\right)^2=\left(\frac{a-y^2}{b}\right)^2,$$

where a and b are two constants.

3. A uniform endless chain is in equilibrium under the action of forces which depend only on the position of the particle acted on. Every element has an equal velocity communicated to it in the direction of the tangent to the chain at that element, prove that the form of the chain will not be altered by the motion.

4. Let a cable be delivered with velocity c' from a ship moving with uniform velocity c in a straight line on the surface of a sea of uniform depth. If the resistance of the water to the cable be proportional to the square of the velocity, the coefficient, B, of resistance for longitudinal motion being different from the coefficient A, for lateral motion, prove that the cable may take the form of a straight line making an angle λ with the horizon, such that $\cot^2\lambda=\sqrt{e^4+\frac{1}{4}}-\frac{1}{2}$, where e is the ratio of the speed of the ship to the terminal velocity of a length of cable falling laterally in water. Prove also that the tension will be found from the equation

$$T=\left\{y-\frac{B}{A}e^2\left(\frac{c'}{c}-\cos\lambda\right)^2\frac{y}{\sin\lambda}\right\}mg'.$$

5. A heavy string is suspended from one extremity, and being slightly disturbed makes small oscillations about the vertical. Find the form of the curve in which it must be placed at rest at the time $t=0$, in order that every point of the string may reach the vertical at the same time.

Result. If the fixed point be the origin and the axis of x be drawn vertically downwards, the equation to the string at the time t is $y=\phi(x)\cos\sqrt{cg}\,t$, where ϕ is to be determined from the equation $(l-x)\dfrac{d^2\phi}{dx^2}-\dfrac{d\phi}{dx}+c\phi=0$, and c is an arbitrary constant depending on the initial form of the curve.

6. The extreme links of a uniform chain can slide freely on two given curves in a vertical plane, and the whole is in equilibrium under the action of gravity. Supposing the chain to break at any point, prove that the initial tension at any point is $T = y(A\phi + B)$, where y is the altitude of the point above the directrix of the catenary, ϕ the angle the tangent makes with the horizon, and A, B two arbitrary constants. Explain how the constants are to be determined.

7. A string is wound round the under part of a vertical circle and is just supported in equilibrium at the ends of a horizontal diameter by two forces. The circle being suddenly removed, prove that the tension at the lowest point is instantly decreased in the ratio $4 : e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}$.

8. Explain in a general manner why the marching of troops in time along a suspension bridge will cause oscillations which may become so great as to be dangerous to the bridge.

In page 485 the different times of vibration of a suspension chain are found. They are such that it is possible that the "marching time" may coincide with some one of them. The result then follows from what is said in page 306.

9. A heavy chain of length $2l$ is suspended from two points A and B in the same horizontal line whose distance apart is not very different from $2l$. If in addition to gravity each element of the chain be acted on by a small vertical force whose magnitude is $f \sin(at + bs)$, show that the corresponding vertical displacements are given by

$$\eta = \frac{f}{gc\,\overline{b^2 - a^2}} \sin(at + bs).$$

Apply this result to show that gusts of wind travelling along a suspension bridge with a velocity nearly equal to $\sqrt{gc}$ may be dangerous to the bridge.

10. A heavy chain of length $2l$ is suspended from two points A, B in the same horizontal line whose distance apart is not very different from $2l$. Each particle of the chain is slightly disturbed from its position of rest in a direction perpendicular to the vertical plane through AB. Find the small oscillations of the chain.

11. A uniform string rests in equilibrium in the form of a circle about a centre of repulsive force in the centre. A small disturbance being given to the string, find the motion.

12. A heavy string is suspended from two fixed points A and B and rests in equilibrium in the form of a catenary whose parameter is c. Let the string be initially displaced, the points of support A, B being also moved, so that

$$\phi = \sigma (1 + \cos 2\alpha) + \sigma' \sin 2\alpha,$$

where σ and σ' are two small quantities and the other letters have the same meaning as in Art. 299. If the string be placed at rest in this new position, prove that it will always remain at rest.

13. An elastic string rests on a a perfectly smooth table, and being attached at its extremities to two fixed points A and B' on the table, is equally stretched throughout. The extremity B' is suddenly released, find the motion of the string.

Referring to Art. 304, we see that when there are no vibrations transversal to the string $ds' = dx'$, and hence the accurate equation of motion is

$$\frac{d^2x'}{dt^2} = a^2 \frac{d^2x'}{dx^2}.$$

The integral of this equation is $x' = f(at - x) + F(at + x)$. The conditions to determine f and F are as follows. Let l be the unstretched and l' the stretched length of the string.

1. When $t = 0$, $x' = nx$ from $x = 0$ to $x = l$, where $n = \frac{l'}{l}$.

2. When $t = 0$, $\frac{dx'}{dt} = 0$ from $x = 0$ to $x = l$.

3. When $x = 0$, $x' = 0$ for all values of t.

4. When $x = l$, $T = 0$; $\therefore \frac{dx'}{dx} = 0$ for all values of t.

From the third condition it follows that the functions F and f are the same with opposite signs. From the fourth condition, we have $f'(at + l) = -f'(at - l)$ so that the values of the function f' recur with opposite signs when the variable is increased by $2l$. The second condition gives $f'(-x) = f'(x)$ and the first gives $f(-x) - f(x) = nx$ from $x = 0$ to $x = l$. Hence $f'(x) = -\frac{n}{2}$ from $x = -l$ to $x = l$. Thus $f(z) = -\frac{n}{2}$ from $z = -l$ to $+l$, and $f(z) = \frac{n}{2}$ from $z = l$ to $3l$ and so on, changing sign every time the variable passes the values l, $3l$, $5l$, &c. Let us consider the motion of any point P of the string whose unstretched abscissa is x. Its velocity is given by the formula $\frac{v}{a} = f'(at-x) - f'(at+x)$. Since $x<l$ we have $\frac{v}{a} = -\frac{n}{2} + \frac{n}{2} = 0$; hence the particle does not move until $at+x=l$. The second function then changes sign and we have $\frac{v}{a} = -\frac{n}{2} - \frac{n}{2} = -n$. The particle continues to move with this velocity until $at-x=l$, when the first function changes sign and so on. Let AB be the unstretched string, and let a point R starting from B move continually along the string and back again with velocity a. Then it is easy to see that when R is on the same side of P as the loose end of the string, P will be at rest, and when R is on same side of P as the fixed end, P will be moving with a velocity alternately equal to $\pm na$. The general character of the motion is; the equilibrium of the string being disturbed at B, a wave of length $4l$ travels along the string, so that P does not begin to move until the wave reaches it. This wave is reflected at A and returns. In this solution we have assumed the equations to hold in the immediate neighbourhood of the points of discontinuity. For a discussion on this point, the reader is referred to treatises on sound.

THE END.

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